MATH 233 Sample Exam #2, Version 1

In problems that require reasoning, algebraic calculation, or the use of your graphing calculator, it is not sufficient just to write the answers. You must explain how you arrived at your answers, show your algebraic calculations, and indicate how you used your graphing calculator.
If approximate numerical answers are used, they should be rounded off to 5 significant figures.

1. Consider the function $f(x,y) = x^2 + xy + y^2$.
(a) (8 points) Find the directional derivative at the point $(1,1)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$. (b) (5 points) Find the unit vector in the direction in which f increases most rapidly at the point $(1,1)$. (c) (7 points) Find the equation of the normal line to the level curve $f(x,y) = x^2 + xy + y^2 = 3$ at the point $(1,1)$.

2. (a) (8 points) Find all local minimum, local maximum values and saddle points of

$$g(x,y) = x^3 + 3xy + y^3.$$

(b) (2 points) Does the function g have *absolute* maximum and minimum values? Justify your answer.
(c) (10 points) Consider the function

$$f(x,y) = x^2 + xy + y^2 - 6x + 2\,.$$

Find its absolute maximum and minimum when $0 \le x \le 5$ and $-3 \le y \le 0$.

3. (a) (15 points) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x,y) = 3x + 4y$, when the points (x,y) lie on the circle $x^2 + y^2 = 1$. (b) (5 points) Sketch the circle $x^2 + y^2 = 1$ and the level curves of $f(x,y) = 3x + 4y$ at the points of maximum and minimum obtained in (a).

4. (20 points) Calculate the integrals:
(a) Given the domain $R = \{(x,y) : 0 \le x \le 2, -1 \le y \le 1\}$,

$$\int\int_R (1 - 6x^2 y)\,dA\,.$$

(b) Find the volume of the solid which lies above the rectangle $-1 \le x \le 1$, $0 \le y \le 1$ and between two surfaces which are graphs of $f(x,y) = 6x^4 + 7x^3y + 4y^2$ and $g(x,y) = x^4 + x^3y + y^2$.

5. (20 points) (a) Find the tangent plane to the surface $z = x^3 + x^2y + xy^2 + y^3$ at the point with x-coordinate -1 and y-coordinate 1.
(b) At the point $(2,1,-3)$ find the tangent plane to the level surface for the function $F(x,y,z) = e^{x+y+z}\sin(x^2 - y + z) - (xy + z)^2$.
(c) Find a vector in the direction of the line of intersection of the tangent planes.

MATH 233 Sample Exam #2, Version 2

In problems that require reasoning, algebraic calculation, or the use of your graphing calculator, it is not sufficient just to write the answers. You must explain how you arrived at your answers, show your algebraic calculations, and indicate how you used your graphing calculator. If approximate numerical answers are used, they should be rounded off to 5 significant figures.

1. (a) (10 points) For the function $f(x,y) = x^4 + 2x^2y^2 - 2y^2 - 1$ find the unit vector in the direction in which f increases most rapidly at the point $(2,-1)$.
(b) What is the maximal directional rate of change and what is the minimal directional rate of change of f at this point?
(c) (10 points) Find the tangent plane to the surface $z = x^4 + 2x^2y^2 - y^2 - 1$ at the point with x-coordinate 2 and y-coordinate -1.

2. (a) (10 points) Find all local minima, local maxima and saddle points of

$$f(x,y) = x^2y^2 + x^2 - 2y^3 + 3y^2$$

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(b) (10 points) Find the absolute maximum and minimum of the function $f(x,y)$ when $-1 \le x \le 1$ and $0 \le y \le 1$.

3. (20 points) Find the minimum and the maximum of the function $f(x,y,z) = y + z - x$ on the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{4} = 1$.

4. (a) (10 points) For the domain $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 2\}$, calculate the following integral *without* a calculator

$$\int\int_R 2x^3 - 3x^2y + 3xy^2 \;\; dA\,.$$

(b) (10 points) Find the volume of the solid bounded by the planes $z = 0$, $x = 0$, $x = 1$, $y = 0$, $y = 1$ and the surface $z = x^4 - 2x^2y^2 + y^4$. (You may use calculators on this problem.) (The last surface lies is above the plane $z = 0$ since $x^4 - 2x^2y^2 + y^4 = (x^2 - y^2)^2 \ge 0$.)

5. Let z be a function of x and y, $z = x^2 + y^2 - xy$, and let $x = x(r,s)$ and $y = y(r,s)$ be functions of r and s, so that we can consider z also as a function of r and s. Suppose that

$$x(r,s) = s - r \text{ and } y(2,3) = 3,\ \frac{\partial y}{\partial r}(2,3) = 7,\ \frac{\partial y}{\partial s}(2,3) = -5\ .$$

(a) (10 points) Calculate $\frac{\partial z}{\partial r}\Big|_{r=2,\ s=3}$ and $\frac{\partial z}{\partial s}\Big|_{r=2,\ s=3}$, the partial derivatives of z with respect to r and s, when $r = 2$ and $s = 3$.
(b) (10 points) Calculate $z\Big|_{r=2,\ s=2}$, and then calculate approximately $z\Big|_{r=2.1,\ s=2.9}$, i.e., the value of z when $r = 2.1$ and $s = 2.9$. (*Hint: use linearization or differential.*)

MATH 233 Sample Exam #2, Version 3

In problems that require reasoning, algebraic calculation, or the use of your graphing calculator, it is not sufficient just to write the answers. You must *explain* how you arrived at your answers, *show* your algebraic calculations, and *indicate* how you used your graphing calculator. If approximate numerical answers are used, they should be rounded off to 5 significant figures.

1. Consider the function $f(x, y) = x^2 + xy + y^2$.
(a) (8 points) Find the directional derivative at the point $(1, 1)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$.
(b) (5 points) Find the unit vector in the direction in which f increases most rapidly at the point $(1, 1)$.
(c) (7 points) Find the equation of the normal line to the level curve $f(x, y) = x^2+xy+y^2 = 3$ at the point $(1, 1)$.

2. (a) (8 points) Find all local minimum values, local maximum values and saddle points of

$$g(x, y) = x^3 + 3xy + y^3.$$

(b) (2 points) Does the function g have *absolute* maximum and minimum values? Justify your answer.
(c) (10 points) Consider the function

$$f(x, y) = x^2 + xy + y^2 - 6x + 2\,.$$

Find its absolute maximum and minimum when $0 \le x \le 5$ and $-3 \le y \le 0$.

3. (a) (15 points) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = 3x + 4y$, when the points (x, y) lie on the circle $x^2 + y^2 = 1$.
(b) (5 points) Sketch the circle $x^2 + y^2 = 1$ and the level curves of $f(x, y) = 3x + 4y$ at the points of maximum and minimum obtained in (a).

4. (20 points) (a) Find the tangent plane to the surface $z = x^3 + x^2y + xy^2 + y^3$ at the point with x-coordinate -1 and y-coordinate 1.
(b) At the point $(2, 1, -3)$ find the tangent plane to the level surface for the function $F(x, y, z) = e^{x+y+z} \sin(x^2 - y + z) - (xy + z)^2$.
(c) Find a vector in the direction of the line of intersection of the tangent planes.

5. (20 points) Consider a machine that is supposed to produce circular cones of radius $r = 2\ m$ and height $h = 6\ m$. The actual size of the cone it produces depends on the temperature T in the room. The correct size is produced at 18^oC and measurements show that when $T = 18^oC$, the rate of change of the radius with respect to temperature is 0.005 $m/^oC$. and the rate of change of the height is 0.012 $m/^oC$. Use the the differentials to estimate the maximal error in the volume of the cone if the temperature in the room is kept between 16^oC and 21^oC.

Math 233 Sample Exam #2, Version 4

1. (a) (10 points) For the function $f(x,y) = x^4 + 2x^2y^2 - 2y^2 - 1$ find the unit vector in the direction in which f increases most rapidly at the point $(2,-1)$.

(b) What is the maximal directional rate of change and what is the minimal directional rate of change of f at this point?

(c) (10 points) Find the tangent plane to the surface $z = x^4 + 2x^2y^2 - y^2 - 1$ at the point with x-coordinate 2 and y-coordinate -1.

2. (a) (10 points) Find all local minima, local maxima and saddle points of

$$f(x,y) = x^2y^2 + x^2 - 2y^3 + 3y^2.$$

(b) (10 points) Find the absolute maximum and minimum of the function $f(x,y)$ when $-1 \leq x \leq 1$ and $0 \leq y \leq 1$.

3. (20 points) Find the minimum and the maximum of the function $f(x,y,z) = y + z - x$ on the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{4} = 1$.

4. Let z be a function of x and y, $z = x^2 + y^2 - xy$, and let $x = x(r,s)$ and $y = y(r,s)$ be functions of r and s, so that we can consider z also as a function of r and s. Suppose that

$$x(r,s) = s - r \text{ and } y(2,3) = 3,\ \frac{\partial y}{\partial r}(2,3) = 7,\ \frac{\partial y}{\partial s}(2,3) = -5\ .$$

(a) (10 points) Calculate $\frac{\partial z}{\partial r}\Big|_{r=2,\ s=3}$ and $\frac{\partial z}{\partial s}\Big|_{r=2,\ s=3}$, the partial derivatives of z with respect to r and s, when $r = 2$ and $s = 3$.

(b) (10 points) Calculate $z\Big|_{r=2,\ s=2}$, and then calculate approximately $z\Big|_{r=2.1,\ s=2.9}$, i.e., the value of z when $r = 2.1$ and $s = 2.9$. (*Hint: use linearization or differential.*)

Math 233 Sample Exam #2, Version 5

1. Let $f(x,y,z) = e^{xyz} + e^{x^2yz} + e^{xy^2z}$.
(a) (3 points) Check that the point $Q = (1,-1,0)$ lies on the surface $f(x,y,z) = 3$.
(b) (6 points) Find an equation of the tangent plane to the surface at Q.
(c) (5 points) Find the rate of change of $f(x,y,z)$ at Q, in the direction from Q to the origin.
(d) (6 points) Find the minimal directional rate of change of $f(x,y,z)$ at Q.

2. Consider a function $f(x,y)$ such that $f(x,y) = \frac{u}{v} + \ln\frac{u}{v}$ where $u = e^{x^2y^2-2xy} + \sin(x-y+1)$ and $v = g(x,y)$ is a function such that $g(1,2) = 1$, $g_x(1,2) = 4$, $g_y(1,2) = 1$.
(a) (10 points) Calculate $\left.\frac{\partial f}{\partial x}\right|_{x=1,\ y=2}$ and $\left.\frac{\partial f}{\partial y}\right|_{x=1,\ y=2}$, the partial derivatives of $f(x,y)$ with respect to x and y, when $x = 1$ and $y = 2$.
(b) (10 points) Calculate approximately $\left.f(x,y)\right|_{x=1.1,\ y=2.1}$ by using a linearization or a differential.

3. (18 points) Use the method of Lagrange multipliers to find the minimum and the maximum of the function $f(x,y) = 12x + 5y$ on the circle $x^2 + y^2 = 169$.

4. (a) (3 points) Show that the point $(2,1)$ is on the curve $\cos(x^3 + y^2 - 9) = y$.
(b) (6 points) Find an equation of the tangent line to the curve $\cos(x^3 + y^2 - 9) = y$ at the point $(2,1)$.

5. (a) (10 points) Find all local minima, local maxima and saddle points of

$$f(x,y) = xy(1 - x + y).$$

(b) (10 points) Find the absolute maximum and minimum of the function $f(x,y)$ in the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,-1)$ (the sides of the triangle are included).

6. (13 points) We are designing a box of height H, width W and depth D. We want it to have volume 2 units. Also, the material used for the four side walls is twice more expensive than the material used for the top and bottom. We want to find the dimensions H, W and D such that the cost of material used to make this box is minimal.
(a) Set up the equations of the Lagrange multiplier method:
(i) write out the function that we want to minimize,
(ii) find the equation of the constraint surface,
(iii) write the function that combines the functions from (i) and (ii) and an auxilliary variable λ.
(b) Find the dimensions H, W and D such that the cost of material used to make this box is minimal.

Math 233 Sample Exam #2, Version 6

1. Let $f(x, y, z) = x^2y^3z^4 + xyz$.
(a) (3 points) Check that the point $Q = (2, 1, -1)$ lies on the surface $f(x, y, z) = 2$.
(b) (6 points) Find an equation of the tangent plane to the surface at Q.
(c) (5 points) Find the rate of change of $f(x, y, z)$ at Q, in the direction from Q to the origin.
(d) (6 points) Find the maximal directional rate of change of $f(x, y, z)$ at Q.

2. Consider the function $z = e^{x^2y^2-2xy} + \sin(x - y + 1)$. If $x = -uv$ and $y = g(u, v)$ is a function such that $g(1, -1) = 2$, $g_x(1, -1) = 3$, $g_y(1, -1) = -2$; then we can consider z as a funcion of u and v.
(a) (10 points) Calculate $\frac{\partial z}{\partial u}$ when $u = 1$ and $v = -1$.
(b) (10 points) Calculate approximately $f(x, y) = \sin(x^2 + y^2 - 9)$ at $x = 2.1$ and $y = 0.9$, by using a linearization or a differential at $x = 2$ and $y = 1$.

3. (18 points) Use the method of Lagrange multipliers to find the minimum and the maximum of the function $f(x, y, z) = x + y + z$ on the ellipsoid $x^2 + 2y^2 + 3z^2 = 66$.

4. (a) (3 points) Show that the point $(2, 1)$ is on the curve $\ln(x^2 + y^2 - 4) = y - 1$.
(b) (6 points) Find an equation of the tangent line to the curve $\ln(x^2 + y^2 - 4) = y - 1$. at the point $(2, 1)$.

5. (a) (10 points) Find all local minima, local maxima and saddle points of

$$f(x, y) = xy(1 - x)(1 - y).$$

(b) (10 points) Find the absolute maximum and minimum of the function $f(x, y)$ in the square with vertices at $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ (the sides of the square are included).

6. (13 points) We are designing an open box (i.e., without the lid), of height z, width x and depth y. We want it to have volume 3 units. Also, the material used for the bottom is twice more expensive than the material used for the four side walls. We want to find the dimensions x, y and z such that the cost of material used to make this box is minimal.
(a) Set up the equations of the Lagrange multiplier method:
(i) write out the function that we want to minimize,
(ii) find the equation of the constraint surface,
(iii) write the function that combines the functions from (i) and (ii) and an auxilliary variable λ.
(b) Find the dimensions x, y and zs such that the cost of material used to make this box is minimal.