Merged-log-concavity of rational functions, almost strictly unimodal sequences, and phase transitions of ideal boson and fermion gases

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March 31, 2025

Abstract

This manuscript introduces the notion of *merged-log-concavity* for rational functions. We then present new results on q-binomial coefficients and unimodal sequences of real values of rational functions, extending q-multinomial coefficients and the Cauchy–Binet formula. The notion is modeled over the q-log-concavity of Stanley for polynomials.

We construct explicit merged-log-concave rational functions, extending the infinite products $(\pm t;q)_\infty^{\mp 1} = \prod_{i\geq 1}(1\pm tq^{i-1})^{\mp 1}$ by polynomials with positive integer coefficients. We derive almost strictly unimodal sequences of rational functions from the merged-log-concavity and Young diagrams. We then study critical points on the almost strictly unimodal sequences. In particular, we obtain the golden ratio of quantum dilogarithms and q-exponentials as a critical point. Also, we consider some eta products and generalized Narayana numbers by the extended q-multinomial coefficients.

In statistical mechanics, we discuss the grand canonical partition functions of certain ideal boson and fermion gases with or without Casimir energies (Ramanujan summation). The merged-log-concavity identifies particle-emergence phase transitions in Helmholtz free energy vacua through critical points such as the golden ratio.

1 Introduction

The notion of unimodal sequences includes increasing, decreasing, or hill sequences of real numbers. As such, the notion is essential in mathematics. Also, there are log-concavities for polynomials [ALGV, BraHuh, But, New, Sag, Sta], which give unimodal sequences. We introduce the notion of *merged-log-concavity* for rational functions. To study the notion, we extend q-multinomial coefficients and the Cauchy–Binet formula. This not only gives q-polynomials with positive integer coefficients, but also the continuous variation of unimodal sequences of real values of rational functions. Furthermore, we obtain the golden ratio and the other metallic ratios as critical points of the variation. These critical points then give statistical mechanical phase transitions of ideal boson and fermion gases.

To give an idea of the merged-log-concavity, we consider $q^{\frac{1}{2}}$-polynomials $f_m(q^{\frac{1}{2}}) = q^{\frac{m^2}{2}}$ of $m \in \mathbb{Z}_{\geq 0}$. They are not $q^{\frac{1}{2}}$-log-concave [But, Sta], since they never give $q^{\frac{1}{2}}$-polynomials with positive integer coefficients by

$$f_m^2 - f_{m-1}^2 f_{m+1}^2 = q^{m^2} - q^{m^2+1}. \tag{1.0.1}$$

However, we obtain the $q^{\frac{1}{2}}$-polynomial with the positive integer coefficient by

$$(1-q)(1-q^2)\left(\left(\frac{f_1}{1-q}\right)^2 - \frac{f_0}{1}\cdot\frac{f_2}{(1-q)(1-q^2)}\right) = q. \tag{1.0.2}$$

We realize q-Pochhammer symbols $(m)_q := \prod_{1\leq i\leq m}(1-q^i)$ in the above. We then introduce the merged-log-concavity for $\mathscr{Q}_m(q^{\frac{1}{2}}) = \frac{f_m}{(m)_q}$, which merges $\mathscr{Q}_1^2 - \mathscr{Q}_0\mathscr{Q}_2$ and $(2)_q$ into q. Also, the merged-log-concavity gives the variation of unimodal sequence $\mathscr{Q}_m(h)$ for $0 < h = q^{\frac{1}{2}} < 1$ in Figure 1. The golden ratio $h = \frac{-1+\sqrt{5}}{2}$ is the critical point of the variation, as it separates hill sequences from strictly decreasing sequences.

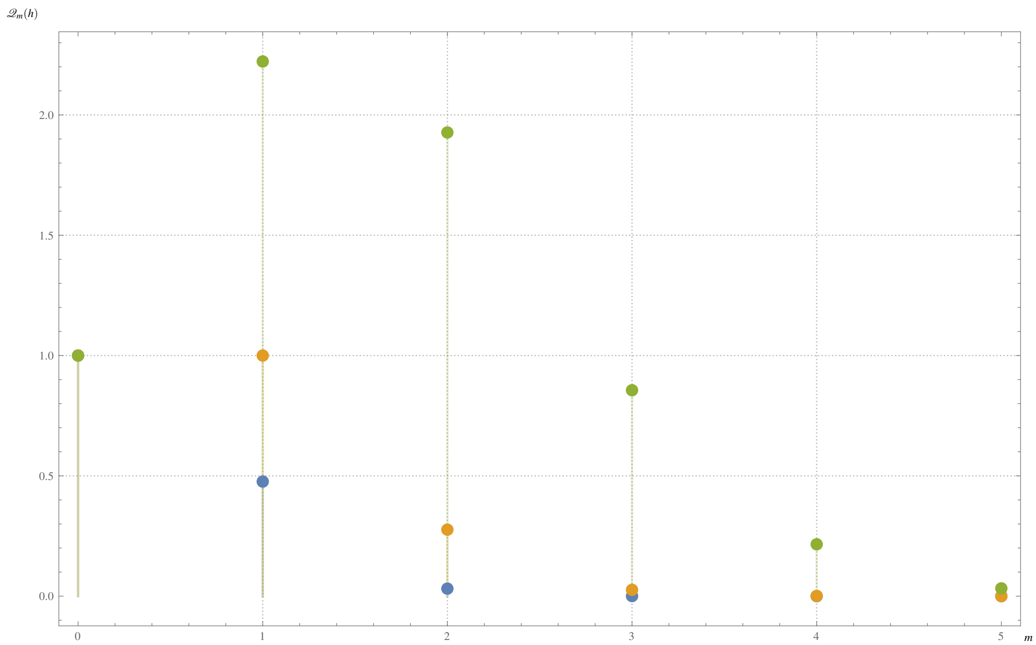


Figure 1: $\mathscr{Q}_m(h)$ of $h = 0.4$ (bottom), $h = \frac{-1+\sqrt{5}}{2}$ (middle), and $h = 0.8$ (top)

The generating function of $\mathscr{F}_m$ gives $(tq^{\frac{1}{2}};q)_\infty^{-1}$ [Eul], i.e.,

$$(tq^{\frac{1}{2}};q)_\infty^{-1} = \prod_{i\geq 1}(1+tq^{\frac{1}{2}}q^{i-1})^{-1} = \sum_{m\geq 0}\mathscr{F}_m t^m.$$

We also obtain the merged-log-concavity for the quantum dilogarithms and q-exponentials $(\pm tq^{\lambda};q)_\infty^{\mp 1}$ of $\lambda \in \mathbb{Q}$, where double-sign corresponds. These $(\pm tq^{\lambda};q)_\infty^{\mp 1}$ have been heavily studied in mathematics and physics. Therefore, the merged-log-concavity provides a framework that generalizes $(\pm tq^{\lambda};q)_\infty^{\mp 1}$ by polynomials with positive integer coefficients and the variation of unimodal sequences. We then discuss explicit examples with open conjectures.

For further discussion, we begin by recalling the fundamental definitions of unimodality and log-concavity for sequences of real numbers.

1.1 Unimodality and log-concavity of real numbers

For $\hat{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$ and $s_1, s_2 \in \hat{\mathbb{Z}}$, let $[\![s_1, s_2]\!] = \{i \in \mathbb{Z} \mid s_1 \leq i \leq s_2\}$ and $[\![s_2]\!] = [\![1, s_2]\!]$ throughout. For example, $[\![\infty]\!] = \mathbb{Z}_{\geq 0}$. Suppose a family $F = (F_i \in U_i)_{i \in I} \in \prod_{i \in I} U_i$ for an index set I and a set U_i. If $I \subset \mathbb{Z}$ is an interval, then F is a sequence of the length $\#(I) \in \hat{\mathbb{Z}}$, i.e., the number of elements in I. If $I = [\![s_2]\!]$ and $s_2 \in \mathbb{Z}_{\geq 1}$, then $F = (F_1, \ldots, F_{s_2})$ is a tuple.

Definition 1.1. *Suppose a sequence $r = (r_i \in \mathbb{R})_{i \in [\![s_1, s_2]\!]}$.*

1. *The sequence r has a step if $r_i < r_j$ for some $i, j \in [\![s_1, s_2]\!]$.*
2. *The sequence r is unimodal if $r_{s_1} \leq \ldots \leq r_\delta \geq r_{\delta+1} \geq \ldots$ for some $\delta \in \hat{\mathbb{Z}}$ such that $s_1 \leq \delta \leq s_2$. In particular, r is a hill if such $\delta \in [\![s_1 + 1, s_2 - 1]\!]$.*
3. *When the sequence r is unimodal, r is two-sided if r has both increasing and decreasing steps, and r is one-sided if r has either increasing steps or decreasing steps, but not both.*
4. *The sequence r is log-concave if $r_i^2 - r_{i+1} r_{i-1} \geq 0$ for each $i \in [\![s_1 + 1, s_2 - 1]\!]$.*

For example, consider an infinite-length $r = (r_i \in \mathbb{R})_{i \in [\![0, \infty]\!]}$. First, suppose that r is unimodal. Then, $\delta = 0$, $\delta \in [\![\infty]\!]$, and $\delta = \infty$ give a decreasing sequence $r_0 \geq r_1 \geq r_2 \geq \ldots$, hill sequence $r_0 \leq \ldots \leq r_\delta \geq r_{\delta+1} \geq \ldots$, and increasing sequence $r_0 \leq r_1 \leq r_2 \leq \ldots$, respectively. We observe that there are one-sided hill sequences such that $r_0 \leq \ldots \leq r_\delta = r_{\delta+1} = \ldots$ and $r_0 < r_\delta$.

Second, suppose that r is log-concave and all its terms are positive. Then, r is unimodal by $\frac{r_1}{r_0} \geq \frac{r_2}{r_1} \geq \ldots$.

1.2 q-log-concavity and strong q-log-concavity of polynomials

Unless stated otherwise, let q be an indeterminate. We recall the q-log-concavity and strong q-log-concavity, on which we model the merged-log-concavity. By q-polynomials $\mathbb{Q}[q]$, Laurent q-polynomials $\mathbb{Q}[q^{\pm 1}]$, and q-rational functions $\mathbb{Q}(q)$, we adopt the following binary relations.

Definition 1.2. *Suppose $f, g \in \mathbb{Q}(q)$.*

1. *Let $f \geq_q g$ if $f, g \in \mathbb{Q}[q]$ and $f - g \in \mathbb{Z}_{\geq 0}[q]$. Also, $f >_q g$ if $f \geq_q g$ and $f - g \neq 0$.*
2. *Let $f \geq_{q^{\pm 1}} g$ if $f, g \in \mathbb{Q}[q^{\pm 1}]$ and $f - g \in \mathbb{Z}_{\geq 0}[q^{\pm 1}]$. Also, $f >_{q^{\pm 1}} g$ if $f \geq_{q^{\pm 1}} g$ and $f - g \neq 0$.*

Stanley and Sagan have introduced the following notions of q-log-concavity and strong q-log-concavity for polynomials [But, Sag]. These notions have been studied intensively [Bre, Sta], but have not been extended to rational functions.

Definition 1.3. *Suppose $f = (f_i(q) \in \mathbb{Z}_{\geq 0}[q])_{i \in \mathbb{Z}}$.*

1. *The sequence f is q-log-concave if $f_i(q)^2 - f_{i-1}(q) f_{i+1}(q) \geq_q 0$ whenever $i \in \mathbb{Z}$.*
2. *The sequence f is strongly q-log-concave if $f_i(q) f_j(q) - f_{i-1}(q) f_{j+1}(q) \geq_q 0$ whenever $j \geq i$.*

Suppose a q-log-concave $(f_i(q) \in \mathbb{Z}_{\geq 0}[q])_{i \in \mathbb{Z}}$ such that $f_i(q) >_q 0$ for each $i \in \mathbb{Z}_{\geq 0}$. Then, we have unimodal $(f_i(r) \in \mathbb{R}_{>0})_{i \in \mathbb{Z}_{\geq 0}}$ for $r \in \mathbb{R}$.

1.3 Merged-log-concavity of rational functions

We introduce the merged-log-concavity of rational functions, using the following q-analogs as foundational elements.

Definition 1.4. *Let a be an indeterminate.*

1. *For each $n \in \hat{\mathbb{Z}}_{\geq 0}$, we have the q-Pochhammer symbol $(a;q)_n = \prod_{i \in [\![n]\!]}(1-aq^{i-1})$ if $n \geq 1$ and $(a;q)_n = 1$ if $n = 0$. Let $(n)_q = (q;q)_n$ in our convention.*
2. *For each $n \in \mathbb{Z}_{\geq 1}$, we have the q-number $[n]_q = \sum_{i \in [\![n]\!]} q^{i-1}$ and q-factorial $[n]!_q = \prod_{i \in [\![n]\!]}[i]_q$. As special cases, $[0]_q = 0$ and $[0]!_q = 1$.*

We use the notation $(n)_q = (q;q)_n$ for the q-Pochhammer symbol, as indices n become involved in several contexts, such as $(1-y_{l-i+1})_q$. Also, the q-Pochhammer symbol notation $(n)_q$ is consistent with the q-number notation $[m]_q$.

We adopt the following notations for family-to-family and family-to-scalar comparisons.

Definition 1.5. *Consider $\lambda \in \mathbb{R} \cup \{\pm\infty\}$. Suppose families $F = (F_i \in \mathbb{R})_{i \in I}$ and $F' = (F'_i \in \mathbb{R})_{i \in I}$.*

1. *Let $F \geq F'$ (or $F > F'$) if each $F_i \geq F'_i$ (or $F_i > F'_i$).*
2. *Let $F \geq \lambda$ (or $F > \lambda$) if each $F_i \geq \lambda$ (or $F_i > \lambda$). Let $F \leq \lambda$ (or $F < \lambda$) if each $F_i \leq \lambda$ (or $F_i < \lambda$).*
3. *Let $F = \lambda$ if $F \leq \lambda$ and $F \geq \lambda$.*

We call F positive if $F > 0$, negative if $F < 0$, and zero if $F = 0$. Similarly, we call F non-negative if $F \geq 0$ and non-positive if $F \leq 0$. Also, let $|F| = (|F_i| \in \mathbb{R}_{\geq 0})_{i \in I}$.

Also, we adopt the following notations for tuples.

Definition 1.6. *Let $d \in \mathbb{Z}_{\geq 1}$.*

1. *Consider sets $V_1, \ldots, V_d$ and $a \in \prod_{i \in [\![d]\!]} V_i$. Then, let $a^\vee = (a^\vee_i = a_{d-i+1})_{i \in [\![d]\!]} = (a_d, \ldots, a_1) \in \prod_{i \in [\![d]\!]} V_{d-i+1}$ for the flip of a.*
2. *Consider a set U and $u \in U$. If $\lambda \subset [\![d]\!]$, then let $\iota^\lambda(u) \in \prod_{i \in \lambda} U$ such that $\iota^\lambda(u)_i = u$ for each $i \in \lambda$. Also, let $\iota^d(u) = \iota^{[\![d]\!]}(u) = (u, \ldots, u) \in U^d = U \times \ldots \times U = \prod_{i \in [\![d]\!]} U$.*

We consider q-numbers and the change of variable $q \mapsto q^\rho$ for $\rho \in \mathbb{Z}_{\geq 1}$. This gives the following.

Definition 1.7. *Let $\lambda \in \mathbb{Z}$, $\rho \in \mathbb{Z}_{\geq 1}$, and $\phi(q) \in \mathbb{Q}(q)$ such that $\phi(q) \neq 0$. We define the base shift function*

$$b(\lambda, \phi, \rho, q) = \begin{cases} \dfrac{\phi(q^\rho)^\lambda [\lambda]!_{q^\rho}}{\phi(q)^\lambda [\lambda]!_q} & \text{if } \lambda \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{otherwise}. \end{cases}$$

In particular, let $b_{\lambda,\rho}(q) = b(\lambda, 1-q, \rho, q)$.

We state the following, as we are interested in q-polynomials with positive integer coefficients.

Lemma 1.8. *Suppose an irreducible $\phi(q) \in \mathbb{Q}[q]$ such that $\phi(0) = 1$. Then, the following statements are equivalent.*

1. *$b(\lambda,\phi,\rho,q) >_q 0$ whenever $\lambda \in \mathbb{Z}_{\geq 0}$ and $\rho \in \mathbb{Z}_{\geq 1}$.*

2. *$\phi(q) = 1-q$.*

When $\phi(q) = 1-q$, we have $b(\lambda,\phi,\rho,q) = b_{\lambda,\rho}(q) = \frac{(\lambda)_{q^\rho}}{(\lambda)_q}$. By these q-Pochhammer symbols, we give the notions of ring shift factors, merged determinants, parcels, and the merged-log-concavity. The ring shift factors extend 2×2 determinants to the merged determinants. We then define the merged-log-concavity of the parcels, which are families of rational functions. Definition 1.9 is a simplified version of Definition 6.1, which allows $\phi(q)$ such as $\phi(q) = 1$ and the merged-log-concavity of parcels of finite positive terms.

Definition 1.9. *Suppose $u^{-1}, \rho, l \in \mathbb{Z}_{\geq 1}$. Also, suppose $w \in \mathbb{Z}^l_{\geq 0}$ and $a,b \in \mathbb{Z}^{2l}$.*

1. *Let $\mathrm{L}(a,b), \mathrm{R}(a,b) \in \mathbb{Z}^l$ such that $\mathrm{L}(a,b)_i = b_i - a_i$ and $\mathrm{R}(a,b)_i = b_{l+i} - a_{l+i}$ for $i \in [\![l]\!]$. Also, for each $\mathscr{F} = (\mathscr{F}_m(q^u) \in \mathbb{Q}(q^u))_{m\in\mathbb{Z}^l}$, let*

$$\det(\mathscr{F},a,b) = \det\begin{bmatrix} \mathscr{F}_{\mathrm{L}(a,b)} & \mathscr{F}_{\mathrm{R}(a^\vee,b)} \\ \mathscr{F}_{\mathrm{L}(a^\vee,b)} & \mathscr{F}_{\mathrm{R}(a,b)} \end{bmatrix}.$$

2. *In $\mathbb{Q}(q^u)$, we define the ring shift factor*

$$\Upsilon(l,w,\rho,a,b,q) = \begin{cases} \displaystyle\prod_{i\in[\![l]\!]} \frac{(b_i)^{w_i}_{q^\rho} (b_{l+i})^{w_{l-i+1}}_{q^\rho}}{(a_i)^{w_i}_{q^\rho} (a_{l+i})^{w_{l-i+1}}_{q^\rho}} \text{ if } a,b \geq 0, \\ 0 \text{ otherwise}. \end{cases}$$

We call ρ the base shift parameter of $\Upsilon(l,w,\rho,a,b,q)$. Furthermore, for each $\mathscr{F} = (\mathscr{F}_m(q^u) \in \mathbb{Q}(q^u))_{m\in\mathbb{Z}^l}$, we define the merged determinant

$$\Delta(\mathscr{F})(l,w,\rho,a,b,q) = \Upsilon(l,w,\rho,a,b,q) \cdot \det(\mathscr{F},a,b) \in \mathbb{Q}(q^u).$$

3. *Suppose a family $f = (f_m \in \mathbb{Q}(q^u))_{m\in\mathbb{Z}^l}$ such that $f_m >_{(q^u)^{\pm 1}} 0$ if $m \geq 0$ and $f_m = 0$ otherwise. Then, we define the parcel $\mathscr{F} = \Lambda(l,w,f,q,u) = (\mathscr{F}_m \in \mathbb{Q}(q^u))_{m\in\mathbb{Z}^l}$ such that*

$$\mathscr{F}_m = \begin{cases} \dfrac{f_m}{\prod_{i\in[\![l]\!]} (m_i)^{w_i}_q} \text{ if } m \geq 0, \\ 0 \text{ otherwise}. \end{cases}$$

We refer to l, w, f, and q as the width, weight, numerator, and base of the parcel $\mathscr{F}$.

4. *For $a,b \in \mathbb{Z}^{2l}$, we call (l,a,b) fitting if it satisfies the following inequalities:*

$$a \leq b;$$
$$b_1 \leq \ldots \leq b_l < b_{l+1} \leq \ldots \leq b_{2l};$$
$$0 \leq a_1 \leq \ldots \leq a_l < a_{l+1} \leq \ldots \leq a_{2l}.$$

Then, we call $\mathscr{F} = \Lambda(l,w,f,q,u)$ ρ-merged-log-concave (or merged-log-concave for simplicity) if each fitting (l,a,b) satisfies

$$\Delta(\mathscr{F})(l,w,\rho,a,b,q) >_{(q^u)^{\pm 1}} 0.$$

On Definition 1.9, we make a few comments.

- A parcel $\mathscr{F}$ is a family of rational functions in $\mathbb{Q}(q^u)$ with parcel parameters. We consider $\mathscr{F} = \mathscr{G}$ for parcels $\mathscr{F}$ and $\mathscr{G}$ when they are the same families of rational functions. However, in most cases, parcel parameters are clear in the context, as we specify them.
- As $\Upsilon(l,w,\rho,a,b,q) = 1$ if $w = 0$, merged determinants generalize 2×2 determinants by ring shift factors. We adopt the term "ring shift factors", since we study $\Delta(\mathscr{F})(l,w,\rho,a,b,q) \in \mathbb{Q}[(q^u)^{\pm 1}]$ by the merged-log-concavity for rational functions $\mathscr{F}_m \in \mathbb{Q}(q^u)$.
- We adopt the term "merged determinants" for $\Delta(\mathscr{F})(l,w,\rho,a,b,q)$, as they merge $\Upsilon(l,w,\rho,a,b,q)$ and $\det(\mathscr{F},a,b)$ into polynomials with positive integer coefficients.

1.4 Monomial indices

By the following notion, we construct explicit merged-log-concave parcels.

Definition 1.10. *Let $l \in \mathbb{Z}_{\geq 1}$, $w \in \mathbb{Z}^l_{\geq 1}$, and $\gamma \in \prod_{i\in[\![l]\!]}\mathbb{Q}^3$. We call $\upsilon = (l,w,\gamma)$ a monomial index if*

$$2\gamma_{i,1} \in \mathbb{Z} \text{ for each } i \in [\![l]\!], \tag{1.4.1}$$

$$0 \leq 2\sum_{j\in[\![i]\!]}\gamma_{j,1} \leq \sum_{j\in[\![i]\!]} w_j \text{ for each } i \in [\![l]\!]. \tag{1.4.2}$$

We call l, w, and γ the width, weight, and core of υ. We refer to (1.4.1) *and* (1.4.2) *as the integer monomial condition and the sum monomial condition of υ.*

We use these monomial indices by the following quadratic polynomials and binary relations.

Definition 1.11. *Suppose $l \in \mathbb{Z}_{\geq 1}$, $\gamma \in \prod_{i\in[\![l]\!]}\mathbb{Q}^3$, and $\alpha \in \mathbb{Z}^l_{\geq 1}$ with $\delta = \gcd(\alpha_1,\ldots,\alpha_l)$.*

1. *Let $t_{\gamma_i}(z) = \gamma_{i,1}z^2 + \gamma_{i,2}z + \gamma_{i,3} \in \mathbb{Q}[z]$ for each $i \in [\![l]\!]$.*
2. *Let $t_{\alpha,\gamma} : \mathbb{Z}^l \to \mathbb{Q}$ such that $t_{\alpha,\gamma}(m) = \sum_{i\in[\![l]\!]}\alpha_i t_{\gamma_i}(m_i)$ for each $m \in \mathbb{Z}^l$.*
3. *Let $u_{\alpha,\gamma} = \frac{\delta}{\lambda} \in \mathbb{Q}$ for the lowest $\lambda \in \mathbb{Z}_{\geq 1}$ such that $\frac{t_{\alpha,\gamma}(m)}{u_{\alpha,\gamma}} \in \mathbb{Z}$ for each $m \in \mathbb{Z}^l_{\geq 0}$.*
4. *Let $\mathfrak{U}_{\alpha,\gamma,q} = \{q^{u_{\alpha,\gamma}}\}$. Then, on $\mathbb{Q}(\mathfrak{U}_{\alpha,\gamma,q})$, let*

$$>_{\alpha,\gamma,q} = >_{q^{u_{\alpha,\gamma}}} \text{ and } \geq_{\alpha,\gamma,q} = \geq_{q^{u_{\alpha,\gamma}}} \text{ if } t_{\alpha,\gamma}(m) \in \mathbb{Q}_{\geq 0} \text{ for each } m \in \mathbb{Z}^d_{\geq 0},$$
$$>_{\alpha,\gamma,q} = >_{q^{\pm u_{\alpha,\gamma}}} \text{ and } \geq_{\alpha,\gamma,q} = \geq_{q^{\pm u_{\alpha,\gamma}}} \text{ otherwise.}$$

For simplicity, let $t_\gamma = t_{\iota^l(1),\gamma}$ and $u_\gamma = u_{\iota^l(1),\gamma}$.

Assume a monomial index (l,w,γ). For each $m \in \mathbb{Z}^l$, let

$$f_{\gamma,m}(q) = \begin{cases} q^{t_\gamma(m)} \text{ if } m \geq 0, \\ 0 \text{ otherwise.} \end{cases}$$

We define *the monomial parcel* $\mathscr{F}_{w,\gamma,q} = \Lambda(l,w,f_\gamma(q),q,u_\gamma)$ such that each $m \in \mathbb{Z}^l_{\geq 0}$ satisfies

$$\mathscr{F}_{w,\gamma,q,m} = \frac{q^{t_\gamma(m)}}{\prod_{i\in[\![l]\!]}(m_i)_q^{w_i}} \in \mathbb{Q}(q^{u_\gamma}).$$

Then, we have the following merged-log-concavity of monomial parcels.

Theorem 1.12 (Theorem 8.40)**.** *Let $\rho \in \mathbb{Z}_{\geq 1}$. Then, every monomial parcel $\mathscr{F}_{w,\gamma,q}$ is ρ-merged-log-concave. In particular, each fitting (l,a,b) satisfies*

$$q^{-t_\gamma(\mathrm{L}(a,b))-t_\gamma(\mathrm{R}(a,b))}\Delta(\mathscr{F}_{w,\gamma,q})(l,w,\rho,a,b,q) >_q 0.$$

Unless stated otherwise, we assume the following simplification in the rest of Section 1:

- $l=1$ and the width of each parcel is l;
- $\mathscr{F}_{\gamma,q} = \mathscr{F}_{(1),\gamma,q}$;
- $\mathscr{F} = \left(\mathscr{F}_m = \mathscr{F}_{(m)} \in \mathbb{Q}(q^u)\right)_{m\in\mathbb{Z}}$ for each parcel $\mathscr{F} = \Lambda(l,w,f,q,u)$.

We deduce $\gamma_{1,1} = \frac{1}{2}$ or 0 for $\mathscr{F}_{\gamma,q}$ by the monomial conditions of $(1,(1),\gamma)$. Consequently, $\mathscr{F}_{\gamma,q}$ give

$$\mathscr{F}_{((0,\gamma_{1,2},\gamma_{1,3})),q,m} = \frac{q^{\gamma_{1,2}m+\gamma_{1,3}}}{(m)_q}, \text{ or}$$

$$\mathscr{F}_{((\frac{1}{2},\gamma_{1,2},\gamma_{1,3})),q,m} = \frac{q^{\frac{m^2}{2}+\gamma_{1,2}m+\gamma_{1,3}}}{(m)_q}.$$

We call $\mathscr{F}_{((0,\gamma_{1,2},\gamma_{1,3})),q}$ and $\mathscr{F}_{((\frac{1}{2},\gamma_{1,2},\gamma_{1,3})),q}$ *$q^{\gamma_{1,2}}$-linear and $q^{\frac{1}{2}+\gamma_{1,2}}$-quadratic* (or *linear and quadratic* for simplicity), since $\frac{q^{\gamma_{1,2}}}{(1)_q} = \frac{\mathscr{F}_{((0,\gamma_{1,2},\gamma_{1,3})),q,1}}{\mathscr{F}_{((0,\gamma_{1,2},\gamma_{1,3})),q,0}}$ and $\frac{q^{\frac{1}{2}+\gamma_{1,2}}}{(1)_q} = \frac{\mathscr{F}_{((\frac{1}{2},\gamma_{1,2},\gamma_{1,3})),q,1}}{\mathscr{F}_{((\frac{1}{2},\gamma_{1,2},\gamma_{1,3})),q,0}}$. In particular, for the $q^{\frac{1}{2}}$-linear and $q^{\frac{1}{2}}$-quadratic monomial parcels, we define

$$\mathscr{L} = \mathscr{F}_{((0,\frac{1}{2},0)),q}, \tag{1.4.3}$$

$$\mathscr{Q} = \mathscr{F}_{((\frac{1}{2},0,0)),q}. \tag{1.4.4}$$

Then, equation (1.0.2) provides a merged determinant of $\mathscr{Q}$ in Theorem 1.12 for $a=(0,1)$ and $b=(1,2)$.

1.5 Parcel convolutions and an extended Cauchy–Binet formula

Consider parcels $\mathscr{F}$ and $\mathscr{G}$ such that multiplying the generating functions of $\mathscr{F}$ and $\mathscr{G}$ becomes the generating function of a parcel. We then discuss the *parcel convolution* $\mathscr{F} * \mathscr{G}$ by the generating functions and Toeplitz matrices of $\mathscr{F}$ and $\mathscr{G}$. Cauchy–Binet formula writes the minors of a matrix product AB by those of A and B. Also, merged determinants extend 2×2 determinants by the ring shift factors.

Therefore, in Theorem 15.11, we extend general minors and the Cauchy–Binet formula of a general matrix product AB to write the merged determinants of $\mathscr{F} * \mathscr{G}$ by those of $\mathscr{F}$ and $\mathscr{G}$. This gives the merged-log-concavity of parcel convolutions.

Furthermore, we define the notion of *multimonomial indices* to consider explicit parcel convolutions.

Definition 1.13. *Let $d \in \mathbb{Z}_{\geq 1}$, $w \in \mathbb{Z}^1_{\geq 1}$, $\alpha,\beta \in \mathbb{Z}^d_{\geq 1}$, and $\gamma \in \prod_{i\in[\![d]\!]}\mathbb{Q}^3$. If $(1,w,(\gamma_i))$ is a monomial index for each $i \in [\![d]\!]$, then we call*

$$\mu = (d,w,\alpha,\beta,\gamma)$$

a multimonomial index. We refer to d, w, α, β, and γ as the depth, weight, inner-exponent, outer-exponent, and core of μ. In particular, if $\beta = \iota^d(1)$, then we call

$$(d,w,\alpha,\gamma)$$

a reduced multimonomial index (or a multimonomial index for short).

We now define the following q-monomials by multimonomial indices.

Definition 1.14. *Consider a multimonomial index (d,w,α,γ). Let*

$$\psi_{\alpha,\gamma,q} = \left(\psi_{\alpha,\gamma,q,j} \in \mathbb{Q}(q^{u_{\alpha,\gamma}})\right)_{j\in\mathbb{Z}^d}$$

such that

$$\psi_{\alpha,\gamma,q,j} = \begin{cases} q^{t_{\alpha,\gamma}(j)} \text{ if } j \geq 0, \\ 0 \text{ otherwise.} \end{cases}$$

Furthermore, we discuss the change of variable $q \mapsto q^\rho$ for $\rho \in \mathbb{Z}_{\geq 1}$. Hence, we define the following *weighted q-multinomial coefficients*, extending q-multinomial coefficients. For $d \in \mathbb{Z}_{\geq 1}$ and $\alpha \in \mathbb{Z}^d_{\geq 1}$, let $\operatorname{lcm}(\alpha)$ and $\gcd(\alpha)$ denote the least common multiple and greatest common divisor of $\alpha_1,\dots,\alpha_d$.

Definition 1.15. *Let $d \in \mathbb{Z}_{\geq 1}$, $\alpha \in \mathbb{Z}^d_{\geq 1}$, and $\delta_l = \operatorname{lcm}(\alpha)$. Suppose $i \in \mathbb{Z}$ and $j \in \mathbb{Z}^d$. Then, we define the weighted q-multinomial coefficient:*

$$\begin{bmatrix} i \\ j \end{bmatrix}_{\alpha,q} = \begin{cases} \dfrac{(i)_{q^{\delta_l}}}{\prod_{\lambda\in[\![d]\!]}(j_\lambda)_{q^{\alpha_\lambda}}} \text{ if } j \geq 0 \text{ and } \displaystyle\sum_{\lambda\in[\![d]\!]} j_\lambda = i, \\ 0 \text{ otherwise.} \end{cases}$$

If $\delta_l = 1$, then let $\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]_{\alpha,q}$ for the q-multinomial coefficient. If $k \in \mathbb{Z}$, then let $\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} i \\ (k,i-k) \end{smallmatrix}\right]_q$ for the q-binomial coefficient.

For the weighted q-multinomial coefficients, we prove the weighted q-Pascal identity. In particular, we have q-polynomials with positive integer coefficients such that $\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]_{\alpha,q} \geq_{q^{\delta_g}} 0$ of $\delta_g = \gcd(\alpha)$.

For example, if $\alpha = (1,2)$, then the weighted q-Pascal identity asserts the following weighted q-Pascal's triangle (written in a rectangular form to save space), where polynomials along arrows indicate multipliers.

$$\begin{array}{ccccc}
\left[\begin{smallmatrix} 0 \\ (0,0) \end{smallmatrix}\right]_{\alpha,q} = 1 & \xrightarrow{1} & \left[\begin{smallmatrix} 1 \\ (0,1) \end{smallmatrix}\right]_{\alpha,q} = 1 & \xrightarrow{1} & \left[\begin{smallmatrix} 2 \\ (0,2) \end{smallmatrix}\right]_{\alpha,q} = 1 \\
\downarrow [2]_q & & \downarrow [2]_q & & \downarrow [2]_q \\
\left[\begin{smallmatrix} 1 \\ (1,0) \end{smallmatrix}\right]_{\alpha,q} = [2]_q & \xrightarrow{q^2} & \left[\begin{smallmatrix} 2 \\ (1,1) \end{smallmatrix}\right]_{\alpha,q} = [4]_q & \xrightarrow{q^2} & \left[\begin{smallmatrix} 3 \\ (1,2) \end{smallmatrix}\right]_{\alpha,q} = [6]_q \\
\downarrow [2]_{q^2} & & \downarrow [2]_{q^2} & & \downarrow [2]_{q^2} \\
\left[\begin{smallmatrix} 2 \\ (2,0) \end{smallmatrix}\right]_{\alpha,q} = [4]_q & \xrightarrow{q^4} & \left[\begin{smallmatrix} 3 \\ (2,1) \end{smallmatrix}\right]_{\alpha,q} = q^7+q^6+2q^5+2q^4+2q^3+2q^2+q+1 & \xrightarrow{q^4} & \left[\begin{smallmatrix} 4 \\ (2,2) \end{smallmatrix}\right]_{\alpha,q} = \dots
\end{array}.$$

By weighted q-multinomial coefficients, the convolutions of monomial parcels give explicit merged-log-concave parcels $\mathscr{F} = \Lambda(l,w,f,q,u)$ such that each $i \in \mathbb{Z}_{\geq 0}$ satisfies

$$\mathscr{F}_i(q^u) = \frac{\sum_{j\in\mathbb{Z}^d} \psi_{\alpha,\gamma,q,j} \left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]^{w_1}_{\alpha,q}}{(i)^{w_1}_{q^\delta}}.$$

Furthermore, the merged determinants of these $\mathscr{F}$ give the following polynomials with positive integer coefficients.

Theorem 1.16. *(Theorem 18.15) Suppose a multimonomial index $\mu = (d, w, \alpha, \gamma)$. Let $\delta_l = \mathrm{lcm}(\alpha)$, $\delta_g = \gcd(\alpha)$, and $\rho \in \mathbb{Z}_{\geq 1}$. For a fitting (l, a, b), consider integers $m = \mathrm{L}(a, b)_1$, $n = \mathrm{R}(a, b)_1$, and $k = \mathrm{R}(a^\vee, b)_1 - n$. Then, we have the following strict positivity:*

$$\begin{aligned}
&\begin{bmatrix} b_1 \\ a_1 \end{bmatrix}_{q^{\delta_l\rho}}^{w_1} \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}_{q^{\delta_l\rho}}^{w_1} b_{m,\rho}(q^{\delta_l})^{w_1} b_{n,\rho}(q^{\delta_l})^{w_1} \sum_{j_1 \in \mathbb{Z}^d} \psi_{\alpha,\gamma,q,j_1} \begin{bmatrix} m \\ j_1 \end{bmatrix}_{\alpha,q}^{w_1} \sum_{j_2 \in \mathbb{Z}^d} \psi_{\alpha,\gamma,q,j_2} \begin{bmatrix} n \\ j_2 \end{bmatrix}_{\alpha,q}^{w_1} \\
&\quad - \begin{bmatrix} b_1 \\ a_2 \end{bmatrix}_{q^{\delta_l\rho}}^{w_1} \begin{bmatrix} b_2 \\ a_1 \end{bmatrix}_{q^{\delta_l\rho}}^{w_1} b_{m-k,\rho}(q^{\delta_l})^{w_1} b_{n+k,\rho}(q^{\delta_l})^{w_1} \sum_{j_1 \in \mathbb{Z}^d} \psi_{\alpha,\gamma,q,j_1} \begin{bmatrix} m-k \\ j_1 \end{bmatrix}_{\alpha,q}^{w_1} \sum_{j_2 \in \mathbb{Z}^d} \psi_{\alpha,\gamma,q,j_2} \begin{bmatrix} n+k \\ j_2 \end{bmatrix}_{\alpha,q}^{w_1} \\
&>_{\alpha,\gamma,q} 0.
\end{aligned}$$

In particular, suppose $\alpha = \iota^d(1)$ of $d \in \mathbb{Z}_{\geq 1}$, $w = \rho = (1)$, $a = (0, 1)$, and $b = (h, h+1)$ of $h \in \mathbb{Z}_{\geq 0}$. Then, the convolutions of linear monomial parcels give the following q-polynomials with positive integer coefficients:

$$[h+1]_q \left(\sum_{j_1 \in \mathbb{Z}^d} \begin{bmatrix} h \\ j_1 \end{bmatrix}_q \right)^2 - [h]_q \left(\sum_{j_1 \in \mathbb{Z}^d} \begin{bmatrix} h+1 \\ j_1 \end{bmatrix}_q \right) \left(\sum_{j_2 \in \mathbb{Z}^d} \begin{bmatrix} h-1 \\ j_2 \end{bmatrix}_q \right) >_q 0.$$

Also, the convolutions of quadratic monomial parcels give the following q-polynomials with positive integer coefficients:

$$\begin{aligned}
&[h+1]_q \left(\sum_{j_1 \in \mathbb{Z}^d} \left(\prod_{i \in [\![d]\!]} q^{\frac{j_{1,i}(j_{1,i}-1)}{2}} \right) \begin{bmatrix} h \\ j_1 \end{bmatrix}_q \right)^2 \\
&\quad - [h]_q \left(\sum_{j_1 \in \mathbb{Z}^d} \left(\prod_{i \in [\![d]\!]} q^{\frac{j_{1,i}(j_{1,i}-1)}{2}} \right) \begin{bmatrix} h+1 \\ j_1 \end{bmatrix}_q \right) \left(\sum_{j_2 \in \mathbb{Z}^d} \left(\prod_{i \in [\![d]\!]} q^{\frac{j_{2,i}(j_{2,i}-1)}{2}} \right) \begin{bmatrix} h-1 \\ j_2 \end{bmatrix}_q \right) >_q 0.
\end{aligned}$$

1.6 Almost strictly unimodal sequences

We recall the following *almost strictly unimodal sequences* [Rea, Section 2.2] and *strictly log-concave sequences*. We consider the variation of almost strictly unimodal sequences under the merged-log-concavity.

Definition 1.17. *Suppose a sequence $r = (r_i \in \mathbb{R})_{i \in [\![s_1, s_2]\!]}$.*

1. *The sequence r is almost strictly unimodal if there exists $\delta \in \hat{\mathbb{Z}}$ with $s_1 \leq \delta \leq s_2$ such that*
   - *$(r_i)_{i \in [\![s_1, \delta]\!]}$ is strictly increasing,*
   - *$r_\delta \geq r_{\delta+1}$, and*
   - *$(r_i)_{i \in [\![\delta+1, s_2]\!]}$ is strictly decreasing.*

   We refer to such δ as the mode of r.
2. *The sequence r is strictly log-concave if $r_i^2 - r_{i-1}r_{i+1} > 0$ for each $i \in [\![s_1+1, s_2-1]\!]$.*

Notice that including the cases $r_\delta = r_{\delta+1}$, we call δ the mode of r for our convenience.

For instance, let $r = (r_i \in \mathbb{R}_{>0})_{i \in [\![s_1, s_2]\!]}$ be strictly log-concave. Then, r is almost strictly unimodal by $\frac{r_{s_1+1}}{r_{s_1}} > \frac{r_{s_1+2}}{r_{s_1+1}} > \dots$. Even when the merged-log-concavity yields strictly log-concave sequences, we often study them as almost strictly unimodal sequences for the following two reasons.

- We discuss the almost strictly unimodal $(\log(r_i) \in \mathbb{R})_{i \in [\![s_1, s_2]\!]}$, which is not necessarily strictly log-concave.
- Each almost strictly unimodal sequence has at most one equation $r_\delta = r_{\delta+1}$, which sits between $r_\delta < r_{\delta+1}$ and $r_\delta > r_{\delta+1}$. Then, as almost strictly unimodal sequences vary, we discuss *critical points* and *phase transitions* in Section 1.6.1.

1.6.1 Critical points and phase transitions

First, let us obtain strictly log-concave sequences by the merged-log-concavity. Let $\mathscr{F} = \Lambda(l, w, f, q, u)$ be ρ-merged-log-concave. Also, suppose $h \in \mathbb{R}$ such that

$$0 < h = q^u < 1, \tag{1.6.1}$$

which gives temperature inequality 1.11.1 in our statistical mechanical discussion later. By inequality (1.6.1), $\frac{1}{1-q}$ exists as a positive real number. Also, by the positivity condition of f in Item 3 in Definition 1.9, we derive the sequence $u(\mathscr{F}, h) = (\mathscr{F}_m(h) \in \mathbb{R}_{>0})_{m \in \mathbb{Z}_{\geq 0}}$.

Furthermore, $u(\mathscr{F}, h)$ is strictly log-concave as follows. Let $m \in \mathbb{Z}_{\geq 0}$. By inequality (1.6.1), we have

$$\Upsilon(l, w, \rho, (0,1), (m, m+1), h^{u^{-1}}) = \left. \frac{(m)_{q^\rho}^{w_1} (m+1)_{q^\rho}^{w_1}}{(0)_{q^\rho}^{w_1} (1)_{q^\rho}^{w_1}} \right|_{q = h^{u^{-1}}} > 0. \tag{1.6.2}$$

Also, the merged-log-concavity yields

$$\Delta(\mathscr{F})(l, w, \rho, (0,1), (m, m+1), q) >_{(q^u)^{\pm 1}} 0. \tag{1.6.3}$$

By inequalities (1.6.2) and (1.6.3), we derive

$$\mathscr{F}_m(h)^2 - \mathscr{F}_{m-1}(h) \mathscr{F}_{m+1}(h) > 0. \tag{1.6.4}$$

More generally, Young diagrams give rise to strictly log-concave sequences in Section 12, where we obtain that the trivial Young diagram $\boxed{1}$ corresponds to the tuples $(m, m+1)$.

Second, let us discuss the critical points and phase transitions. Among almost strictly unimodal sequences such as $u(\mathscr{F}, h)$, there are sequences that are hill sequences and decreasing sequences simultaneously. If $u(\mathscr{F}, h)$ is one of these boundary sequences, then we call h a critical point of $\mathscr{F}$. In particular, since $u(\mathscr{F}, h)$ is almost strictly unimodal, $u(\mathscr{F}, h)$ is a hill and decreasing sequence if and only if there is a critical point h such that

$$\mathscr{F}_0(h) = \mathscr{F}_1(h). \tag{1.6.5}$$

As for the term "critical points", these h give the zero discrete derivatives $0 = \frac{\mathscr{F}_1(h) - \mathscr{F}_0(h)}{1-0}$ for the discrete variable $m \in \mathbb{Z}_{\geq 0}$ of $u(\mathscr{F}, h) = (\mathscr{F}_m(h) \in \mathbb{R}_{>0})_{m \in \mathbb{Z}_{\geq 0}}$.

Also, we say that $\mathscr{F}$ has a phase transition if along $0 < h < 1$, $u(\mathscr{F}, h)$ changes from a strictly decreasing sequence to a two-sided hill or strictly increasing sequence. This phase transition passes through equation (1.6.5).

1.7 Metallic ratios as explicit critical points

For $n \in \mathbb{Z}_{\geq 1}$, it is well-known that $\frac{-n+\sqrt{n^2+4}}{2} : 1 = 1 : \frac{n+\sqrt{n^2+4}}{2}$. Based on this proportionality, we call the following real numbers

$$\frac{-n+\sqrt{n^2+4}}{2} < 1$$

metallic ratios in our convention, instead of $\frac{n+\sqrt{n^2+4}}{2} > 1$ in [GilWor, Section 1]. For instance, $\frac{-1+\sqrt{5}}{2} = 0.618\ldots$ and $\frac{-2+\sqrt{8}}{2} = 0.414\ldots$ are the golden and silver ratios.

In particular, the golden ratio $h = \frac{-1+\sqrt{5}}{2}$ is the critical point of the $q^{\frac{1}{2}}$-linear monomial parcel $\mathscr{L}$ in equation (1.4.3), since $h = q^{\frac{1}{2}} = q^{u\left(0,\frac{1}{2},0\right)}$ solves

$$\mathscr{L}_0(q^{\frac{1}{2}}) = 1 = \frac{q^{\frac{1}{2}}}{1-q} = \mathscr{L}_1(q^{\frac{1}{2}}).$$

Therefore, $\mathscr{L}$ has the phase transition at the golden ratio in Figure 2 (see Figure 1 for $\mathscr{Q}$).

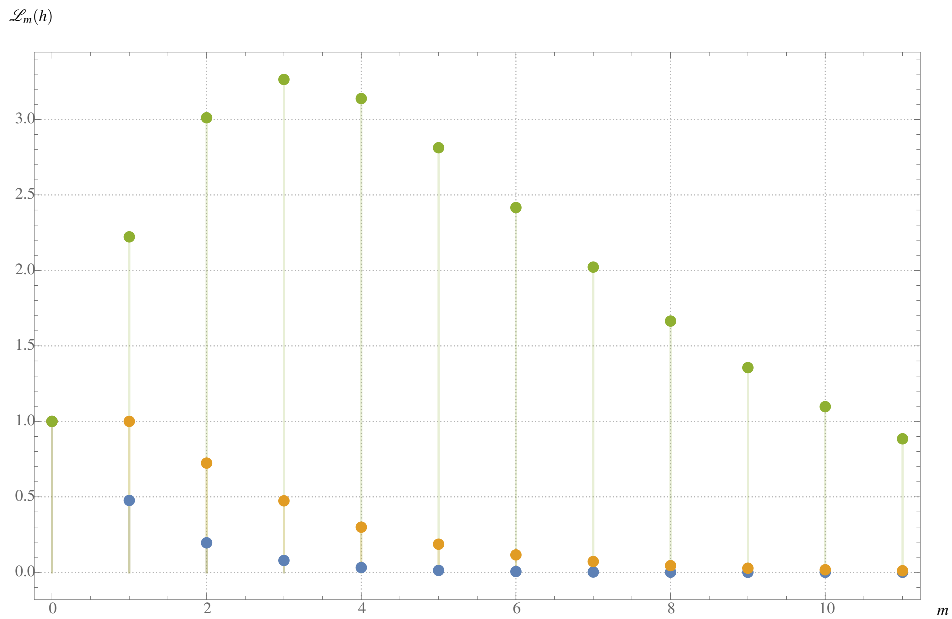


Figure 2: $\mathscr{L}_m(h)$ of $h = 0.4$ (bottom), $h = \frac{-1+\sqrt{5}}{2}$ (middle), and $h = 0.8$ (top)

Furthermore, we consider the n-fold convolutions $\mathscr{L}^{*n}$ for $n \in \mathbb{Z}_{\geq 1}$. Then, all the critical points of $\mathscr{L}^{*n}$ are precisely the metallic ratios, as they solve

$$\mathscr{L}_0^{*n}(q^{\frac{1}{2}}) = 1 = \frac{nq^{\frac{1}{2}}}{1-q} = \mathscr{L}_1^{*n}(q^{\frac{1}{2}})$$

for $0 < q^{\frac{1}{2}} < 1$. For the $q^{\frac{1}{2}}$-quadratic monomial parcel $\mathscr{Q}$ in equation (1.4.4), the same applies to the n-fold convolutions $\mathscr{Q}^{*n}$, since $\mathscr{Q}_0 = \mathscr{L}_0$ and $\mathscr{Q}_1 = \mathscr{L}_1$.

1.8 Characterizations of $\mathscr{L}$ and $\mathscr{Q}$

As we are interested in q-polynomials, we introduce the following notion.

Definition 1.18. *Let $\rho \in \mathbb{Z}_{\geq 1}$. Also, let $\mathscr{F} = \Lambda(l,w,f,q,u)$. Then, we call $\mathscr{F}$ ρ-ideal if $\mathscr{F}$ is ρ-merged-log-concave and each $m \in \mathbb{Z}_{\geq 0}$ satisfies*

$$\Delta(\mathscr{F})(l,w,\rho,(0,1),(m,m+1),q) >_q 0.$$

Section 17.2 verifies that if a monomial parcel $\mathscr{F}_{\gamma,q}$ is ρ-ideal for some $\rho \in \mathbb{Z}_{\geq 1}$, then $\mathscr{F}_{\gamma,q}$ is ρ'-ideal for any $\rho' \in \mathbb{Z}_{\geq 1}$. Hence, we simply call $\mathscr{F}_{\gamma,q}$ ideal if it is ρ-ideal for some $\rho \in \mathbb{Z}_{\geq 1}$.

Consider all the ideal monomial parcels $\mathscr{F}_{\gamma,q}$. Then, the $q^{\frac{1}{2}}$-linear $\mathscr{L}$ is extremal among them by phase transitions as follows. If $\mathscr{F}_{\gamma,q}$ has a phase transition, then in Section 17.2, we prove that $\mathscr{L}_m$ is the maximum among the values $\frac{\mathscr{F}_{\gamma,q,m}}{\mathscr{F}_{\gamma,q,0}}$ in $\mathbb{R}$ for each $m \in \mathbb{Z}_{\geq 0}$ and $0 < q < 1$; i.e.,

$$\mathscr{L}_m = \frac{\mathscr{L}_m}{\mathscr{L}_0} = \frac{q^{\frac{m}{2}}}{(m)_q} \geq \frac{q^{\gamma_{1,1}m^2+\gamma_{1,2}m}}{(m)_q} = \frac{\mathscr{F}_{\gamma,q,m}}{\mathscr{F}_{\gamma,q,0}} \tag{1.8.1}$$

for each $m \in \mathbb{Z}_{\geq 0}$ and $0 < q < 1$. Likewise, the $q^{\frac{1}{2}}$-quadratic $\mathscr{Q}$ is extremal among all the ideal quadratic $\mathscr{F}_{\gamma,q}$ that have phase transitions.

We now adopt the following notation for generating functions.

Definition 1.19. *Let t be an indeterminate. For each $\mathscr{F} = \Lambda(l,w,f,q,u)$, we define the generating function*

$$\mathscr{Z}_{\mathscr{F}}(t) = \sum_{m \in \mathbb{Z}_{\geq 0}} \mathscr{F}_m(q^u)t^m.$$

In particular, let $\mathscr{Z}_{w,\gamma,q}(t) = \mathscr{Z}_{\mathscr{F}_{w,\gamma,q}}(t)$ and $\mathscr{Z}_{\gamma,q}(t) = \mathscr{Z}_{\mathscr{F}_{\gamma,q}}(t)$.

1.8.1 The golden ratio of quantum dilogarithms and q-exponentials

We recall the following equations [Eul, Chapter 16], which also hold for $q,t \in \mathbb{C}$ such that $|q|,|t| < 1$.

Definition 1.20. *Assume the ring of formal power series $\mathbb{Q}[[q,t]]$. Then, we call the following the Euler binomial identities:*

$$(t;q)_\infty^{-1} = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{t^\lambda}{(\lambda)_q};$$
$$(-t;q)_\infty = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{q^{\frac{\lambda(\lambda-1)}{2}}}{(\lambda)_q} t^\lambda.$$

For simplicity, suppose $\mathscr{F}_{\gamma,q,0} = 1$. Then, $\gamma_{1,3} = 0$. Since $\gamma_{1,1} = 0$ or $\gamma_{1,1} = \frac{1}{2}$ by the monomial conditions, we have

$$\mathscr{Z}_{\gamma,q}(t) = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{q^{\gamma_{1,1}\lambda^2+\gamma_{1,2}\lambda}}{(\lambda)_q} t^\lambda = \begin{cases} (tq^{\gamma_{1,2}};q)_\infty^{-1} = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{q^{\gamma_{1,2}\lambda}}{(\lambda)_q} t^\lambda \text{ if } \gamma_{1,1} = 0, \\ (-tq^{\frac{1}{2}+\gamma_{1,2}};q)_\infty^{-1} = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{q^{\frac{\lambda^2}{2}+\gamma_{1,2}\lambda}}{(\lambda)_q} t^\lambda \text{ if } \gamma_{1,1} = \frac{1}{2}. \end{cases}$$

Hence, $\mathscr{Z}_{\gamma,q}(t)$ are *quantum dilogarithms* [FadKas, FadVol, Kir, KonSoi, Rom, Schu, Zag], as they satisfy pentagon identities (see Section 17.1). They have been studied intensively, but to our knowledge, the golden ratio of quantum dilogarithms has not been obtained in the literature. Also, $(\pm t;q)_\infty^{\mp 1}$ are q-extensions of the exponential function [KoeSwa, equations (0.7.7) and (0.7.8)], since

$$e^t = \lim_{q \to 1^-} (\pm(1-q)t;q)_\infty^{\mp 1} = \begin{cases} \lim_{q \to 1^-} ((1-q)t;q)_\infty^{-1}, \\ \lim_{q \to 1^-} (-(1-q)t;q)_\infty. \end{cases} \tag{1.8.2}$$

It follows that $\mathscr{Z}_{\gamma,q}(t)$ are not only quantum dilogarithms, but also q-exponentials.

Therefore, we obtain the golden ratio of the quantum dilogarithms and q-exponentials $\mathscr{Z}_{\gamma,q}(t)$ as the critical point of $\mathscr{L}$, since the $q^{\frac{1}{2}}$-linear $\mathscr{L}$ is the single extremal parcel among $\mathscr{F}_{\gamma,q}$ and the critical point of $\mathscr{L}$ is the golden ratio as in Section 1.8.

Remark 1.21. By Lemma 1.8, we are often interested in $\mathscr{F}_{\gamma,q}$. However, let $\mathscr{G}_{\gamma,q} = \left(\mathscr{G}_{\gamma,q,m} \in \mathbb{Q}(q^{u_\gamma})\right)_{m\in\mathbb{Z}}$ such that

$$\mathscr{G}_{\gamma,q,m} = \begin{cases} \dfrac{q^{\gamma_{1,1}m^2+\gamma_{1,2}m}}{[m]!_q} & \text{if } m \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, these $\mathscr{Z}_{\mathscr{G}_{\gamma,q}}(t)$ are quantum dilogarithms and q-exponentials, which are also often studied.

Furthermore, $\mathscr{G}_{\gamma,q}$ are merged-log-concave in the general merged-log-concavity of Definition 6.1, which allows $[m]!_q$ for $(m)_q$ in Item 3 of Definition 1.9. Then, just like $u(\mathscr{F}_{\gamma,q},h)$, $u(\mathscr{G}_{\gamma,q},h)$ are almost strictly unimodal sequences for $0 < h = q^{u_\gamma} < 1$.

However, unlike $u(\mathscr{F}_{\gamma,q},h)$, $u(\mathscr{G}_{\gamma,q},h)$ have no phase transitions for $0 < h = q^{u_\gamma} < 1$, since h does not satisfy

$$1 = \mathscr{G}_{\gamma,q,0} = \mathscr{G}_{\gamma,q,1} = q^{\gamma_{1,1}+\gamma_{1,2}}.$$

Therefore, even with $\mathscr{G}_{\gamma,q}$, the golden ratio still emerges as the critical point of the quantum dilogarithms and q-exponentials $\mathscr{Z}_{\gamma,q}(t)$ and $\mathscr{Z}_{\mathscr{G}_{\gamma,q}}(t)$, since the $q^{\frac{1}{2}}$-linear $\mathscr{L}$ is the unique extremal parcel among $\mathscr{F}_{\gamma,q}$ and its critical point is precisely the golden ratio.

1.9 Comparison with the strong q-log-concavity and q-log-concavity

First, we compare the strong q-log-concavity and the merged-log-concavity. By Definition 1.9, we focus on positive merged determinants and infinite positive terms in Section 1. However, let $s_1, s_2 \in \hat{\mathbb{Z}}$. Then, a strongly q-log-concave $f = (f_m(q) \in \mathbb{Z}_{\geq 0}[q])_{m\in\mathbb{Z}}$ such that $f_m >_q 0$ for $m \in [\![s_1, s_2]\!]$ is a weight-zero merged-log-concave parcel by the full merged-log-concavity in Definition 6.1. We explain this in more detail in Section 11.

Also, we introduce Hadamard (term-wise) products of parcels for the merged-log-concavity. Consequently, if there is a strongly q-log-concave $f = (f_m(q) \in \mathbb{Z}_{\geq 0}[q])_{m\in\mathbb{Z}}$ such that $f_m >_q 0$ for $m \geq 0$ and $f_m = 0$ for $m < 0$, then we obtain the weight-one merged-log-concave parcel $\mathscr{F}$ such that each $m \geq 0$ satisfies

$$\mathscr{F}_m = \frac{f_m}{(m)_q}.$$

However, the converse is not true, since the numerators $q^{\frac{m(m-1)}{2}}$ of $\mathscr{F}_{((\frac{1}{2},-\frac{1}{2},0)),q}$ are not strongly q-log-concave by equation (1.0.1).

Second, we compare the q-log-concavity and the merged-log-concavity. This gives q-log-concave polynomials from weight-zero merged-log-concave parcels. Also, a merged-log-concave parcel deduces the strict analog of q-log-concavity (1.6.3), which in turn gives almost strictly unimodal sequences of our interest.

1.9.1 On Newton's log-concavities

More explicitly, we compare the q-log-concavity and the merged-log-concavity by finite or infinite geometric sequences.

For finite geometric sequences, we recall the following *Newton's log-concavities* on polynomials. For $d \in \mathbb{Z}_{\geq 2}$ and $\alpha = (\alpha_\lambda \in \mathbb{R}_{>0})_{\lambda \in [\![d]\!]}$, let $p_\alpha = \left(p_{\alpha,\lambda} \in \mathbb{R}_{>0}\right)_{\lambda \in [\![0,d]\!]}$ such that

$$\prod_{i \in [\![d]\!]} (1 + \alpha_i t) = \sum_{\lambda \in [\![0,d]\!]} p_{\alpha,\lambda} t^\lambda.$$

Then, Newton claimed $\frac{\lambda}{\lambda+1}\frac{d-\lambda}{d+1-\lambda} p_{\alpha,\lambda}^2 - p_{\alpha,\lambda-1} p_{\alpha,\lambda+1} \geq 0$ for all $\lambda \in [\![d-1]\!]$ [New, p241–p243] (see [Bra, Lemma 1.1] for a proof). In particular, since $\frac{\lambda}{\lambda+1}\frac{d-\lambda}{d+1-\lambda} < 1$, all $\lambda \in [\![d-1]\!]$ satisfy

$$p_{\alpha,\lambda}^2 - p_{\alpha,\lambda-1} p_{\alpha,\lambda+1} \geq 0,$$

which we call Newton's log-concavity of α. This gives the unimodality of p_α such as the unimodality of binomial coefficients p_α when $\alpha = \iota^d(1)$.

First, let $0 < h < 1$. For $d \in \mathbb{Z}_{\geq 2}$, we consider the finite geometric sequences

$$g(h,d) = \left(h^\lambda\right)_{\lambda \in [\![0,d]\!]}.$$

The q-log-concavity of $(-t;q)_d$ extends Newton's log-concavity of $g(h,d)$ as follows. By the q-binomial theorem, consider $\mathscr{B}(d,\lambda,q) \in \mathbb{Q}[q]$ such that

$$(-t;q)_d = \sum_{\lambda \in [\![0,d]\!]} q^{\frac{\lambda(\lambda-1)}{2}} \begin{bmatrix} d \\ \lambda \end{bmatrix}_q t^\lambda = \sum_{\lambda \in [\![0,d]\!]} \mathscr{B}(d,\lambda,q) t^\lambda.$$

Then, [Sag, Theorem 3.2] (see [But, Kra]) gives the q-log-concavity:

$$\mathscr{B}(d,\lambda,q)^2 - \mathscr{B}(d,\lambda-1,q)\mathscr{B}(d,\lambda+1,q) \geq_q 0. \tag{1.9.1}$$

This inequality implies Newton's log-concavity of $g(h,d)$, when we substitute $q = h$.

Second, we extend to the following infinite case by the infinite geometric sequences

$$g(h,\infty) = \left(h^\lambda\right)_{\lambda \in [\![0,\infty]\!]}.$$

By the Euler binomial identity, we have

$$(-t;q)_\infty = \mathscr{Z}_{((\frac{1}{2},-\frac{1}{2},0)),q}(t).$$

Also, the merged-log-concavity in Theorem 1.12 and the ideal property for $\mathscr{F}_{((\frac{1}{2},-\frac{1}{2},0)),q}$ give

$$\Delta(\mathscr{F}_{((\frac{1}{2},-\frac{1}{2},0)),q})(l,w,\rho,a,b,q) >_q 0$$

for each fitting (l,a,b) and $\rho \in \mathbb{Z}_{\geq 1}$. Therefore, this inequality extends Newton's log-concavity of the finite geometric sequences $g(h,d)$ to the infinite geometric sequences $g(h,\infty)$ via the above q-polynomials with positive integer coefficients.

We have $(t;q)_\infty^{-1} = \mathscr{Z}_{((0,0,0)),q}(t)$ by the Euler binomial identity. We also introduce separable products of parcels, which correspond to multiplying suitable $\mathscr{Z}_{\mathscr{F}}(t)$ and $\mathscr{Z}_{\mathscr{G}}(t')$ for distinct indeterminates t and t'. Hence, the merged-log-concavity generalizes the quantum dilogarithms and the q-exponentials $(\pm t;q)_\infty^{\mp}$ by monomial parcels with convolutions and Hadamard/separable products.

1.10 Monomial convolutions and eta products

We recall the eta function and eta products as follows. These functions, along with exponentials, have been studied intensively [HeiNeu, HonZha, Koh, NekOko].

Definition 1.22. *For the imaginary unit* i*, let $q = e^{2\pi \mathrm{i} \tau}$ of $\tau \in \mathbb{C}$ such that $\mathrm{Im}(\tau) > 0$. Then, we have the (Dedekind) eta function:*

$$\eta(\tau) = q^{\frac{1}{24}}(q;q)_\infty.$$

Also, let $d \in \mathbb{Z}_{\geq 1}$, $\alpha \in \mathbb{Z}_{\geq 1}^d$, and $\beta \in \mathbb{Z}_{\neq 0}^d$. Then, we have the eta product:

$$E_{d,\alpha,\beta}(\tau) = \prod_{\lambda \in [\![d]\!]} \eta(\alpha_\lambda \tau)^{\beta_\lambda}.$$

Weight-one linear and quadratic monomial parcels specialize to the eta function $\eta(\tau)$ and exponential. Hence, we introduce the following notions of *graded monomial products* and *monomial convolutions* by the generating functions of monomial parcels. Then, merged-log-concave parcels give (q,t)-analogs of exponentials and eta products for some tuple t of indeterminates.

Definition 1.23. *Suppose a multimonomial index $(d,w,\alpha,\beta,\gamma)$.*

1. *Let $z = (z_i)_{i \in [\![d]\!]}$ be a tuple of indeterminates. Then, we define the graded monomial product*

$$M(d,w,\alpha,\beta,\gamma,q,z) = \prod_{\lambda \in [\![d]\!]} \left(\mathscr{Z}_{\mathscr{F}_{w,(\gamma_\lambda),q^{\alpha_\lambda}}}(z_\lambda) \right)^{\beta_\lambda} \in \mathbb{Q}(q^{u_{\alpha,\gamma}})[[z_1,\dots,z_d]].$$

2. *Let v be an indeterminate. Then, we define the monomial convolution*

$$\mathscr{M}(d,w,\alpha,\beta,\gamma,q,v) = M(d,w,\alpha,\beta,\gamma,q,\iota^d(v)) = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \mathscr{M}(d,w,\alpha,\beta,\gamma,q,v)_\lambda v^\lambda.$$

We call $M(d,w,\alpha,\beta,\gamma,q,z)$ graded, because it is graded as formal power series of z. For each λ, the monomial convolutions $\left(\mathscr{Z}_{\mathscr{F}_{w,(\gamma_\lambda),q^{\alpha_\lambda}}}(z_\lambda) \right)^{\beta_\lambda}$ correspond to parcel convolutions. Furthermore, multiplying $\left(\mathscr{Z}_{\mathscr{F}_{w,(\gamma_\lambda),q^{\alpha_\lambda}}}(z_\lambda) \right)^{\beta_\lambda}$ in $M(d,w,\alpha,\beta,\gamma,q,z)$ for different λ corresponds to separable products of parcels.

Also, when $w=(1)$, $\mathscr{F}_{w,(\gamma_\lambda),q^{\alpha_\lambda}}$ are linear and quadratic monomial parcels, which satisfy $\gamma_{\lambda,1} = 0$ or $\gamma_{\lambda,1} = \frac{1}{2}$ with arbitrary $\gamma_{\lambda,2}, \gamma_{\lambda,3} \in \mathbb{Q}$. Hence, we introduce the following notation to obtain the (q,t)-analogs.

Definition 1.24. *Let $d \in \mathbb{Z}_{\geq 1}$, $\alpha \in \mathbb{Z}^d_{\geq 1}$, $\beta \in \mathbb{Z}^d_{\neq 0}$, and $\kappa \in \mathbb{Q}^d$. Suppose a tuple $z = (z_\lambda)_{\lambda \in [\![d]\!]}$ of indeterminates.*

1. *Let $\gamma(\beta,\kappa) \in \prod_{\lambda \in [\![d]\!]} \mathbb{Q}^3$ such that*

$$\gamma(\beta,\kappa)_\lambda = \begin{cases} \left(0, \kappa_\lambda, -\frac{1}{24}\right) & \text{if } \beta_\lambda \leq -1, \\ \left(\frac{1}{2}, \frac{\kappa_\lambda}{2}, \frac{1}{24}\right) & \text{otherwise}. \end{cases}$$

2. *Let $T(z,q,\alpha,\beta,\kappa) = (T(z,q,\alpha,\beta,\kappa)_\lambda)_{\lambda \in [\![d]\!]}$ such that*

$$T(z,q,\alpha,\beta,\kappa)_\lambda = \begin{cases} q^{-(1-\kappa_\lambda)\alpha_\lambda} \cdot z_\lambda & \text{if } \beta_\lambda \leq -1, \\ -q^{-\frac{1-\kappa_\lambda}{2}\alpha_\lambda} \cdot z_\lambda & \text{otherwise}. \end{cases}$$

In particular, let $w = (1)$ and $t = T(z,q,\alpha,\beta,\kappa)$. Also, let $q = e^{2\pi i \tau}$ for some $\tau \in \mathbb{C}$ such that $\mathrm{Im}(\tau) > 0$. Then, as $z \in \mathbb{C}^d$ varies, the dominated convergence theorem implies

$$E_{d,\alpha,\beta}(\tau) = \lim_{t \to \iota^d(1)} M(d,w,\alpha,|\beta|,\gamma(\beta,\kappa),q,z). \tag{1.10.1}$$

Hence, $M(d,w,\alpha,|\beta|,\gamma(\beta,\kappa),q,z)$ is the (q,t)-analog of

- the exponential $e^{\sum_{\lambda \in [\![d]\!]} \alpha_\lambda \beta_\lambda z_\lambda}$ when $q \to 1^-$ by equation (1.8.2), and
- the eta product $E_{d,\alpha,\beta}(\tau)$ when $t \to \iota^d(1)$ by equation (1.10.1).

We prove that all graded monomial convolutions and monomial convolutions are generating functions of merged-log-concave parcels, where the parcels of graded monomial convolutions are of general widths. In particular, we obtain the merged-log-concavity of the (q,t)-analogs $M(d,w,\alpha,|\beta|,\gamma(\beta,\kappa),q,z)$ and the monomial convolutions $\mathscr{M}(d,w,\alpha,|\beta|,\gamma(\beta,\kappa),q,v)$. Then, merged determinants give polynomials with positive integer coefficients by the weighted q-multimonomial coefficients in Theorem 1.16. Furthermore, we give conjectures on these polynomials with positive integer coefficients. For example, let $\lambda \in \mathbb{Z}_{\geq 0}$. Then, in Conjecture 18.17, we conjecture a new log-concavity for the trivial eta product

$$1 = \eta(\tau)^\lambda \cdot \eta(\tau)^{-\lambda},$$

which becomes non-trivial by the (q,t)-analogs. Also, we conjecture a periodicity on the polynomials with positive integer coefficients by the generalized Narayana numbers [Guy].

1.11 Statistical-mechanical phase transitions by the merged-log-concavity

Bose, Einstein, and Fermi [Bos, Ein, Fer] pioneered the mathematical models of non-interacting particles as ideal boson and fermion gases. Sections 1.11 and 19 consider the *grand canonical partition functions* of some ideal boson, fermion, or (mixed) boson–fermion gases by monomial convolutions introduced in Section 1.10, since the grand canonical partition functions of these ideal boson and fermion systems coincide with monomial convolutions in Definition 1.23.

We observe the vacua that have the lowest *Helmholtz free energies* in these ideal boson and fermion systems at different temperatures. For this purpose, we examine the grand canonical partition functions by the merged-log-concavity introduced in Section 1.3, which provides a mathematical framework to analyze certain generating functions of rational functions by polynomials. Then, we obtain statistical-mechanical phase transitions on vacua. In particular, zero particle vacua transition to non-zero particle vacua when the temperature crosses critical thresholds, which are determined by the golden ratio and other metallic ratios. Unlike Bose-Einstein condensations, particle numbers in the vacuum determined by the golden ratio continue to increase as the temperature increases. The golden ratio emerges from geometric properties of almost strictly unimodal sequences obtained by the merged-log-concavity, where the almost strictly unimodal sequences realize sequences of Helmholtz free energies.

More explicitly, we discuss the following.

- Section 1.11 considers the grand canonical partition functions of the ideal boson or fermion gases by monomial parcels in Section 1.4. In particular, we discuss the grand canonical partition functions of the extremal $\mathscr{L}$ and $\mathscr{Q}$, which are the $q^{\frac{1}{2}}$-linear and $q^{\frac{1}{2}}$-quadratic monomial parcels in Section 1.8.
- Section 19 considers the grand canonical partition functions of the boson–fermion gases with or without Casimir energies (Ramanujan summation of zero-point energies) by monomial convolutions, which generalize monomial parcels.

Unless stated otherwise, Section 1.11 assumes the thermodynamic beta $\beta > 0$ and chemical potential $\mu < 0$ with $q = e^{-\beta}$, $\mu' = -\mu\beta > 0$, and $t = e^{-\mu'}$, where t represents the fugacity. Then, we have

$$0 < q, t < 1. \tag{1.11.1}$$

The above $0 < q < 1$ is a temperature inequality, since the temperature T satisfies

$$\beta = \frac{1}{Tk_B} \tag{1.11.2}$$

for the Boltzmann constant $k_B = 1.380649 \times 10^{-23}\,\mathrm{J}\cdot\mathrm{K}^{-1}$ by the Joule and Kelvin units J and K. In particular, the temperature inequality $0 < q < 1$ gives mathematical inequality (1.6.1).

For background on the fundamental concepts of statistical mechanics in this manuscript, the reader is referred to [KapGal, Chapter 1].

1.11.1 Ideal boson gases

Let $\delta_{\lambda,\lambda'}$ for $\lambda, \lambda' \in \mathbb{Q}$ denote the Kronecker delta function such that $\delta_{\lambda,\lambda'} = 1$ if $\lambda = \lambda'$ and $\delta_{\lambda,\lambda'} = 0$ otherwise. We consider the following operators and numbers to describe ideal boson gases.

Definition 1.25. *Suppose $\lambda \in \mathbb{Z}_{\geq 1}$ and $v \in \mathbb{Q}$.*

1. *Let $a_{b,\lambda}$ and $a^{\dagger}_{b,\lambda}$ be the boson annihilation and creation operators that satisfy the following commutator relations:*

$$[a_{b,\lambda}, a^{\dagger}_{b,\lambda'}] = \delta_{\lambda,\lambda'};$$
$$[a^{\dagger}_{b,\lambda}, a^{\dagger}_{b,\lambda'}] = [a_{b,\lambda}, a_{b,\lambda'}] = 0.$$

2. *Let*

$$\varepsilon_{v,\lambda} = \lambda - v \in \mathbb{Q}.$$

Also, let $H_{b,v}$ and N_b be the Hamiltonian and number operators such that

$$\begin{aligned} H_{b,v} &= \sum_{\lambda\in\mathbb{Z}_{\geq 1}} \varepsilon_{v,\lambda} a^{\dagger}_{b,\lambda} a_{b,\lambda}, \\ N_b &= \sum_{\lambda\in\mathbb{Z}_{\geq 1}} a^{\dagger}_{b,\lambda} a_{b,\lambda}. \end{aligned} \tag{1.11.3}$$

3. *Let $\gamma_{b,v} = ((0, 1-v, 0)) \in \mathbb{Q}^3$.*

Then, the boson system $B(1,v)$ defined by $H_{b,v}$ and N_b represents an ideal boson gas. The boson system $B(1,v)$ has the grand canonical partition function

$$\mathscr{Z}_{B(1,v)}(q,t) = \operatorname{Tr}\left(e^{-\beta\left(H_{b,v}-\mu N_b\right)}\right) = \operatorname{Tr}\left(e^{-\beta H_{b,v}} \cdot e^{-\mu' N_b}\right)$$

such that

$$\mathscr{Z}_{B(1,v)}(q,t) = \mathscr{Z}_{\gamma_{b,v},q}(t), \tag{1.11.4}$$

where $\mathscr{Z}_{\gamma_{b,v},q}(t)$ is the generating function of the monomial parcel $\mathscr{F}_{\gamma_{b,v},q}$ in Definition 1.19.

Let us obtain the t-expansion of $\mathscr{Z}_{B(1,v)}(q,t)$ (known in physics [Dim, Chapter 1]). We have the eigenvalues $n_\lambda \in \mathbb{Z}_{\geq 0}$ of $a^{\dagger}_{b,\lambda} a_{b,\lambda}$ for the eigenvectors $|n_\lambda\rangle = \frac{1}{\sqrt{n_\lambda!}}(a^{\dagger}_{b,\lambda})^{n_\lambda}|0\rangle$, where n_λ indicates the occupation number of a state λ. Then, the system has the boson Fock space with basis vectors $|n_1, n_2, \ldots, n_k, \ldots\rangle$ such that $\sum_{\lambda\in\mathbb{Z}_{\geq 1}} n_\lambda < \infty$ for states with finitely many particles. In particular, for $\lambda \in \mathbb{Z}_{\geq 1}$ and $n_\lambda \in \mathbb{Z}_{\geq 0}$,

$$\langle n_1, n_2, \ldots, n_k, \ldots | e^{-\beta H_{b,v}} \cdot e^{-\mu' N_b} | n_1, n_2, \ldots, n_k, \ldots \rangle = e^{-\beta \sum_{\lambda\in\mathbb{Z}_{\geq 1}} n_\lambda \varepsilon_{v,\lambda}} \cdot e^{-\mu' \sum_{\lambda\in\mathbb{Z}_{\geq 1}} n_\lambda}.$$

The Euler binomial identity in Definition 1.20 gives equation (1.11.4), since multiplying the above over all states $\lambda \in \mathbb{Z}_{\geq 1}$, we obtain

$$\mathscr{Z}_{B(1,v)}(q,t) = \prod_{\lambda\in\mathbb{Z}_{\geq 1}} \sum_{n_\lambda\in\mathbb{Z}_{\geq 0}} e^{-\beta n_\lambda \varepsilon_{v,\lambda}} \cdot e^{-\mu' n_\lambda} = \prod_{\lambda\in\mathbb{Z}_{\geq 1}} \sum_{n_\lambda\in\mathbb{Z}_{\geq 0}} (q^{\lambda - v} \cdot t)^{n_\lambda} = (tq^{1-v}; q)^{-1}_{\infty} = \mathscr{Z}_{\gamma_{b,v},q}(t).$$

For each $n \in \mathbb{Z}_{\geq 1}$, suppose that $B(n,v)$ has n sub-systems with negligible interactions and $B(1,v)$ represents each sub-system. This system $B(n,v)$ describes an ideal boson gas with the grand canonical partition function $\mathscr{Z}_{\gamma_{b,v},q}(t)^n$.

1.11.2 Ideal fermion gases

Similarly, we consider the following operators and numbers for ideal fermion gases by the energies $\varepsilon_{v,\lambda}$ in Definition 1.25.

Definition 1.26. *Assume $\lambda \in \mathbb{Z}_{\geq 1}$ and $v \in \mathbb{Q}$.*

1. *Let $a_{f,\lambda}$ and $a^{\dagger}_{f,\lambda}$ be the fermion annihilation and creation operators that satisfy the anti-commutator relations:*

$$\begin{aligned} \{a_{f,\lambda}, a^{\dagger}_{f,\lambda'}\} &= \delta_{\lambda,\lambda'}; \\ \{a^{\dagger}_{f,\lambda}, a^{\dagger}_{f,\lambda'}\} &= \{a_{f,\lambda}, a_{f,\lambda'}\} = 0. \end{aligned}$$

2. *Let $H_{f,v}$ and N_f be the Hamiltonian and number operators such that*

$$H_{f,v} = \sum_{\lambda \in \mathbb{Z}_{\geq 1}} \varepsilon_{v,\lambda} a^{\dagger}_{f,\lambda} a_{f,\lambda},$$
$$N_f = \sum_{\lambda \in \mathbb{Z}_{\geq 1}} a^{\dagger}_{f,\lambda} a_{f,\lambda}.$$

3. *Let $\gamma_{f,v} = \left(\left(\frac{1}{2}, \frac{1}{2} - v, 0\right)\right) \in \mathbb{Q}^3$.*

We obtain the fermion system $F(1,v)$ of $H_{f,v}$ and N_f, which is of an ideal fermion gas. This system $F(1,v)$ has the grand canonical partition function

$$\mathscr{Z}_{F(1,v)}(q,t) = \mathrm{Tr}\left(e^{-\beta H_{f,v}} \cdot e^{-\mu' N_f}\right)$$

such that

$$\mathscr{Z}_{F(1,v)}(q,t) = \mathscr{Z}_{\gamma_{f,v},q}(t). \tag{1.11.5}$$

Let us derive equation (1.11.5). Let $\lambda \in \mathbb{Z}_{\geq 1}$. Then, $n_\lambda \in \{0,1\}$ are the eigenvalues of $a^{\dagger}_{f,\lambda} a_{f,\lambda}$ for the eigenvectors $|0\rangle$ and $a^{\dagger}_{f,\lambda}|0\rangle$ by the Pauli exclusion principle. This gives the fermion Fock space with basis vectors $|n_1, n_2, \ldots, n_k, \ldots\rangle$ such that $\sum_{\lambda \in \mathbb{Z}_{\geq 1}} n_\lambda < \infty$. It follows that

$$1 + q^{\lambda - v} \cdot t = \sum_{n_\lambda \in \{0,1\}} e^{-\beta n_\lambda \varepsilon_{v,\lambda}} \cdot e^{-\mu' n_\lambda},$$
$$\langle n_1, n_2, \ldots, n_k, \ldots | e^{-\beta H_{f,v}} \cdot e^{-\mu' N_f} | n_1, n_2, \ldots, n_k, \ldots \rangle = e^{-\beta \sum_{\lambda \in \mathbb{Z}_{\geq 1}} n_\lambda \varepsilon_{v,\lambda}} \cdot e^{-\mu' \sum_{\lambda \in \mathbb{Z}_{\geq 1}} n_\lambda},$$

Since $\mathscr{Z}_{\gamma_{f,v},q}(t) = \sum_{r \in \mathbb{Z}_{\geq 0}} \frac{q^{\frac{r^2}{2} + \left(\frac{1}{2} - v\right) r}}{(r)_q} t^r = (-tq^{1-v}; q)_\infty$ by the Euler binomial identity, equation (1.11.5) follows. [1].

Assume that for each $n \in \mathbb{Z}_{\geq 1}$, $F(n,v)$ has n sub-systems with negligible interactions and $F(1,v)$ represents each sub-system. Consequently, $F(n,v)$ represents an ideal fermion gas with the grand canonical partition function $\mathscr{Z}_{\gamma_{f,v},q}(t)^n$.

1.11.3 Extremal $\mathscr{L}$ and $\mathscr{Q}$ on free energies

Consider the t-series coefficients of $\mathscr{Z}_{\gamma_{b,v},q}(t)^n$ and $\mathscr{Z}_{\gamma_{f,v},q}(t)^n$:

$$\mathscr{Z}_{\gamma_{b,v},q}(t)^n = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} Z_{f,v,n,\lambda}(q^{u\gamma_{b,v}}) t^\lambda;$$
$$\mathscr{Z}_{\gamma_{f,v},q}(t)^n = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} Z_{F,v,n,\lambda}(q^{u\gamma_{f,v}}) t^\lambda.$$

These $Z_{f,v,n,\lambda}(q^{u\gamma_{b,v}})$ and $Z_{f,v,n,\lambda}(q^{u\gamma_{b,v}})$ of particle numbers λ are the *canonical partition functions* of $B(n,v)$ and $F(n,v)$. In particular, $n \in \mathbb{Z}_{\geq 1}$ and $\lambda \in \mathbb{Z}_{\geq 0}$ give $Z_{f,v,n,\lambda}(q^{u\gamma_{b,v}}) \in \mathbb{R}_{>0}$ and $Z_{f,v,n,\lambda}(q^{u\gamma_{b,v}}) \in \mathbb{R}_{>0}$ by

[1] This equation (1.11.5) does not require $|t| = |e^{-\mu'}| < 1$ [Koh, Lemma 1.2]. But, we assume $\mu' > 0$, i.e., $\mu < 0$, for simplicity. See also [Cow, Fig 1 and 2] for μ in high temperatures.

inequality (1.11.1). We then recall the Helmholtz free energies (or free energies for short) $A_{b,v,n,\lambda}(q^{u_{\gamma_{b,v}}})$ and $A_{f,v,n,\lambda}(q^{u_{\gamma_{f,v}}})$ of the canonical partition functions such that

$$A_{b,v,n,\lambda}(q^{u_{\gamma_{b,v}}}) = -\frac{\log(Z_{f,v,n,\lambda}(q^{u_{\gamma_{b,v}}}))}{\beta},$$
$$A_{f,v,n,\lambda}(q^{u_{\gamma_{f,v}}}) = -\frac{\log(Z_{f,v,n,\lambda}(q^{u_{\gamma_{f,v}}}))}{\beta}.$$

By inequality (1.6.4), the free energies satisfy the following inequalities of real numbers:

$$2A_{b,v,n,\lambda}(q^{u_{\gamma_{b,v}}}) - \sum_{i\in\{-1,1\}} A_{b,v,n,\lambda+i}(q^{u_{\gamma_{b,v}}}) < 0; \tag{1.11.6}$$
$$2A_{f,v,n,\lambda}(q^{u_{\gamma_{f,v}}}) - \sum_{i\in\{-1,1\}} A_{f,v,n,\lambda+i}(q^{u_{\gamma_{f,v}}}) < 0. \tag{1.11.7}$$

The merged determinants of monomial convolutions decompose energy inequalities (1.11.6) and (1.11.7) into Laurent $q^{u_{\gamma_{b,v}}}$- or $q^{u_{\gamma_{f,v}}}$-polynomials in Theorem 1.16.

In particular, suppose ideal monomial parcels $\mathscr{F}_{\gamma_{b,v},q}$ and $\mathscr{F}_{\gamma_{f,v},q}$ that have phase transitions. Then, the merged determinants decompose the inequalities into q-polynomials. Furthermore, by inequality (1.8.1), $A_{b,\frac{1}{2},1,\lambda}(q^{\frac{1}{2}})$ of the $q^{\frac{1}{2}}$-linear monomial parcel $\mathscr{L}$ are the lowest among $A_{b,v,1,\lambda}(q^{u_{\gamma_{b,v}}})$ and $A_{f,v,1,\lambda}(q^{u_{\gamma_{f,v}}})$; i.e., $0<q<1$ and $\lambda \in \mathbb{Z}_{\geq 0}$ satisfy

$$A_{b,\frac{1}{2},1,\lambda}(q^{\frac{1}{2}}) \leq A_{b,v,1,\lambda}(q^{u_{\gamma_{b,v}}}),$$
$$A_{b,\frac{1}{2},1,\lambda}(q^{\frac{1}{2}}) \leq A_{f,v,1,\lambda}(q^{u_{\gamma_{f,v}}}).$$

Also, $A_{f,\frac{1}{2},1,\lambda}(q^{\frac{1}{2}})$ of the $q^{\frac{1}{2}}$-quadratic monomial parcel $\mathscr{Q}$ satisfy the same inequalities to $A_{f,v,1,\lambda}$. Therefore, $\mathscr{L}$ and $\mathscr{Q}$ are extremal on the free energies $A_{b,v,1,\lambda}(q^{u_{\gamma_{b,v}}})$ and $A_{f,v,1,\lambda}(q^{u_{\gamma_{f,v}}})$, where the corresponding $\mathscr{F}_{\gamma_{b,v},q}$ and $\mathscr{F}_{\gamma_{f,v},q}$ are ideal monomial parcels that have phase transitions.

1.11.4 Phase transitions

We consider the n-fold convolutions $\mathscr{L}^{*n}$ and $\mathscr{Q}^{*n}$ of the extremal $\mathscr{L}$ and $\mathscr{Q}$. Then, we have

$$\mathscr{Z}_{B(n,\frac{1}{2})}(q,t)^n = \mathscr{Z}_{\mathscr{L}}(t)^n = \mathscr{Z}_{\mathscr{L}^{*n}}(t),$$
$$\mathscr{Z}_{F(n,\frac{1}{2})}(q,t)^n = \mathscr{Z}_{\mathscr{Q}}(t)^n = \mathscr{Z}_{\mathscr{Q}^{*n}}(t).$$

Also, since $u_{\gamma_{b,\frac{1}{2}}} = u_{\gamma_{f,\frac{1}{2}}} = \frac{1}{2}$, let

$$A_{\mathscr{L},n}(q^{\frac{1}{2}}) = \left(A_{\mathscr{L},n,\lambda}(q^{\frac{1}{2}}) := A_{b,\frac{1}{2},n,\lambda}(q^{\frac{1}{2}})\right)_{\lambda\in\mathbb{Z}_{\geq 0}},$$
$$A_{\mathscr{Q},n}(q^{\frac{1}{2}}) = \left(A_{\mathscr{Q},n,\lambda}(q^{\frac{1}{2}}) := A_{f,\frac{1}{2},n,\lambda}(q^{\frac{1}{2}})\right)_{\lambda\in\mathbb{Z}_{\geq 0}}.$$

Then, since $\beta>0$, the positivities in Theorem 1.16 yield the following phase transitions on these free energies by critical points in the mathematical sense of Section 1.6.

Corollary 1.27. *(Corollary 17.21 by the terminology of Section 1.11) Let $n \in \mathbb{Z}_{\geq 1}$.*

1. *Then, we have the almost strictly unimodal sequences $-A_{\mathscr{L},n}(q^{\frac{1}{2}})$ by the critical point $0 < c_{\mathscr{L},n} < 1$ and modes $m_{\mathscr{L},n}(q^{\frac{1}{2}})$ as follows.*
   (a) *For each $0 < q^{\frac{1}{2}} < c_{\mathscr{L},n}$, $m_{\mathscr{L},n}(q^{\frac{1}{2}}) = 0$ gives the strictly decreasing sequence:*
   $$-A_{\mathscr{L},n,m_{\mathscr{L},n}(q^{\frac{1}{2}})}(q^{\frac{1}{2}}) > -A_{\mathscr{L},n,1}(q^{\frac{1}{2}}) > -A_{\mathscr{L},n,2}(q^{\frac{1}{2}}) > \dots.$$
   (b) *If $q^{\frac{1}{2}} = c_{\mathscr{L},n}$, then $m_{\mathscr{L},n}(q^{\frac{1}{2}}) = 0$ gives the hill and decreasing sequence:*
   $$-A_{\mathscr{L},n,m_{\mathscr{L},n}(q^{\frac{1}{2}})}(q^{\frac{1}{2}}) = -A_{\mathscr{L},n,1}(q^{\frac{1}{2}}) > -A_{\mathscr{L},n,2}(q^{\frac{1}{2}}) > \dots.$$
   (c) *For each $1 > q^{\frac{1}{2}} > c_{\mathscr{L},n}$, $m_{\mathscr{L},n}(q^{\frac{1}{2}}) \in \mathbb{Z}_{\geq 1}$ gives the two-sided hill sequence:*
   $$-A_{\mathscr{L},n,0}(q^{\frac{1}{2}}) < \cdots < -A_{\mathscr{L},n,m_{\mathscr{L},n}(q^{\frac{1}{2}})}(q^{\frac{1}{2}}) \geq -A_{\mathscr{L},n,m_{\mathscr{L},n}(q^{\frac{1}{2}})+1}(q^{\frac{1}{2}}) > \dots.$$
2. *We have the same for the almost strictly unimodal sequences $-A_{\mathscr{Q},n}(q^{\frac{1}{2}})$ by the critical point $0 < c_{\mathscr{Q},n} = c_{\mathscr{L},n} < 1$ and modes $m_{\mathscr{Q},n}(q^{\frac{1}{2}})$.*
3. *In particular, we have the metallic ratios*
   $$c_{\mathscr{L},n} = c_{\mathscr{Q},n} = \frac{-n+\sqrt{n^2+4}}{2},$$
   which is the golden ratio for $n = 1$.

By equation (1.11.2), as the temperature T increases, Corollary 1.27 implies particle-emergence phase transitions in the free energy vacua:

$$A_{\mathscr{L},n,m_{\mathscr{L},n}(q^{\frac{1}{2}})}(q^{\frac{1}{2}}) = \min(A_{\mathscr{L},n}(q^{\frac{1}{2}}));$$
$$A_{\mathscr{Q},n,m_{\mathscr{L},n}(q^{\frac{1}{2}})}(q^{\frac{1}{2}}) = \min(A_{\mathscr{Q},n}(q^{\frac{1}{2}})).$$

Specifically, a low temperature T such that $0 < q^{\frac{1}{2}} \leq c_{\mathscr{L},n}$ gives the zero particle mode

$$m_{\mathscr{L},n}(q^{\frac{1}{2}}) = m_{\mathscr{Q},n}(q^{\frac{1}{2}}) = 0,$$

while a high temperature T such that $1 > q^{\frac{1}{2}} > c_{\mathscr{L},n}$ gives the non-zero particle mode

$$m_{\mathscr{L},n}(q^{\frac{1}{2}}), m_{\mathscr{Q},n}(q^{\frac{1}{2}}) \geq 1.$$

In particular, the temperatures T_n of $c_{\mathscr{L},n} = c_{\mathscr{Q},n}$ are

$$T_n = \frac{1}{-2\log\left(\frac{-n+\sqrt{n^2+4}}{2}\right)\cdot(1.38\cdots\times 10^{-23})}K$$

such that

$$T_1 = 0.75\cdots\times 10^{23}K,$$
$$T_2 = 0.41\cdots\times 10^{23}K,$$
$$\cdots$$

There are critical points even for lower temperatures, because we have critical points on $B(n,v)$ and $F(n,v)$ of $n\in\mathbb{Z}_{\geq 1}$ and $v\in\mathbb{Q}_{<1}$, solving

$$nq^{1-\kappa} = 1-q$$

for $0<q^{1-\kappa}<1$.

We now reinterpret Figure 2. Figure 3 illustrates the phase transition of the boson free energies $A_{\mathscr{L},1}(q^{\frac{1}{2}})$ with the zero and non-zero particle vacua by the golden ration $c_{\mathscr{L},1}$. We have a similar figure for the fermion free energies $A_{\mathscr{Q},1}(q^{\frac{1}{2}})$.

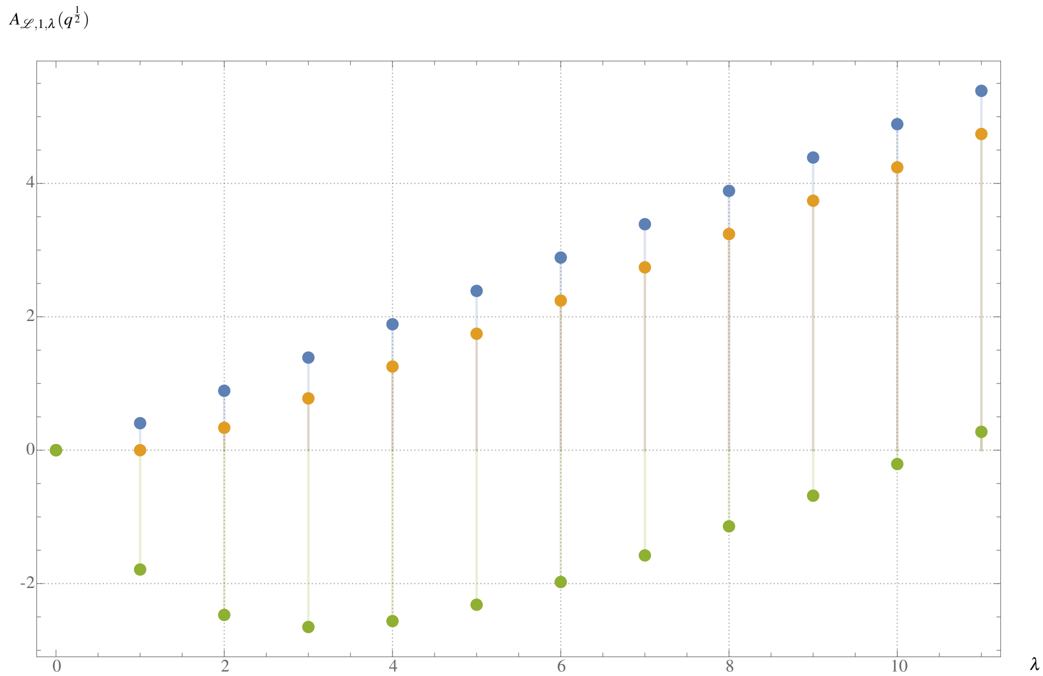


Figure 3: $A_{\mathscr{L},1,\lambda}(q^{\frac{1}{2}})$ of $q^{\frac{1}{2}}=0.4$ (top), $q^{\frac{1}{2}}=c_{\mathscr{L},1}$ (middle), and $q^{\frac{1}{2}}=0.8$ (bottom)

Bose-Einstein condensations differ from the phase transitions in Figure 3, because non-zero particle vacua keep appearing for higher temperatures. More precisely, for each $\lambda\in\mathbb{Z}_{\geq 1}$, we obtain the λ-particle vacuum at the temperature that solves

$$q^{\frac{1}{2}} = 1-q^{\lambda}.$$

It is natural to consider many states at high temperatures. But, instead of the infinite product $(-tq^{\frac{1}{2}};q)_\infty$, there is the following finite analog to the phase transition on $A_{\mathscr{Q},1}(q^{\frac{1}{2}})$. Consider

$$(-tq^{\frac{1}{2}};q)_2 = 1 + (q^{\frac{1}{2}} + q^{\frac{3}{2}})t + q^2t^2 = \sum_{i \in [\![0,2]\!]} F(q^{\frac{1}{2}})_i t^i.$$

Then, $F(q^{\frac{1}{2}}) = \left(F(q^{\frac{1}{2}})_i \in \mathbb{R}_{>0}\right)_{i \in [\![0,2]\!]}$ for each $0 < q^{\frac{1}{2}} < 1$ is strictly log-concave by the discriminant of the t-polynomial $(-tq^{\frac{1}{2}};q)_2$ (c.f. inequality (1.9.1)). We have the critical point $c = 0.68233\dots$ that solves $1 = q^{\frac{1}{2}} + q^{\frac{3}{2}}$. Therefore, $F(q^{\frac{1}{2}})$ is a strictly decreasing sequence for $0 < q^{\frac{1}{2}} < c$, a hill and decreasing sequence for $q^{\frac{1}{2}} = c$, and a two-sided hill sequence for $c < q^{\frac{1}{2}} < 1$.

Acknowledgments

The author thanks Professors Tom Braden, Francesco Brenti, Bruno Carneiro da Cunha, Sergey Galkin, Seikou Kato, Anatol N. Kirillov, Yoshinori Mishiba, Aníbal L. S. Netto, Takuya Okuda, Bruce E. Sagan, Fernando A. N. Santos, and Richard P. Stanley for their helpful communications. Among them, Professor Takuya Okuda encouraged the author to study much further at a very early stage of this work. Also, Professor Francesco Brenti responded to the author in an email at a very early stage of this work, suggesting several references. This work was supported by the Research Institute for Mathematical Sciences, a joint usage/research center located in Kyoto University. In particular, the author thanks Professor Anatol N. Kirillov for reading preliminary versions of this manuscript and providing helpful comments in Kyoto during the author's two several-day visits in 2017 and 2018. The author thanks Departamento de Matemática of Universidade Federal de Pernambuco in Recife for their excellent support to his stay from April 2018 to March 2019.

2 Notations for families, rings, and some q-analogs

We fix some notations to develop our theory of the merged-log-concavity.

2.1 Families

Definition 2.1. *Suppose a family $F = (F_i \in U)_{i \in I}$ of some set U. Then, F is flat if $F_i = F_j$ whenever $i, j \in I$. Let $\mathrm{L}(F) = \#(I) \in \hat{\mathbb{Z}}$. Also, for $u \in U$, let $\mathrm{L}_u(F) = \#(\{i \in I \mid F_i = u\})$.*

Also, we adopt the following notation of families by rings.

Definition 2.2. *Let R be a commutative ring. Consider families $F = (F_i \in R)_{i \in I}$ and $F' = (F'_i \in R)_{i \in I}$.*

1. *Let $F^{F'} = \left(F_i^{F'_i} \in R\right)_{i \in I}$ whenever every $F_i^{F'_i} \in R$ is defined.*
2. *Let $F \pm F' = (F_i \pm F'_i)_{i \in I}$.*
3. *Let $F \circ F' = (F_i F'_i)_{i \in I}$ for the Hadamard product (term-wise product).*
4. *For a scalar $\lambda \in R$, suppose $F'' = (\lambda)_{i \in I}$. Then, let $\lambda \pm F = F \pm \lambda = F'' \pm F$ and $\lambda F = F\lambda = F'' \circ F$.*
5. *Let $\sum F = \sum_{i \in I} F_i$ and $\prod F = \prod_{i \in I} F_i$ whenever they are defined.*

We define the following tuples by increasing integers.

Definition 2.3. *Suppose $d \in \mathbb{Z}_{\geq 1}$ and $\lambda \in \mathbb{Z}^2$.*

1. *Let* $\mathrm{T}_{<}(d,\lambda) = \left\{m \in \mathbb{Z}^d \mid \lambda_1 \leq m_1 < \cdots < m_d \leq \lambda_2\right\}$ *and* $\mathrm{T}_{<}(d,\lambda_2) = \mathrm{T}_{<}(d,(1,\lambda_2))$.

2. *Let* $\mathrm{T}_{\leq}(d,\lambda) = \left\{m \in \mathbb{Z}^d \mid \lambda_1 \leq m_1 \leq \ldots \leq m_d \leq \lambda_2\right\}$ *and* $\mathrm{T}_{\leq}(d,\lambda_2) = \mathrm{T}_{\leq}(d,(1,\lambda_2))$.

We adopt the following notation by tuples. For our convenience, let $[\![s]\!] = [\![(s_1,s_2)]\!] = [\![s_1,s_2]\!]$ if $s \in \hat{\mathbb{Z}}^2$,

Definition 2.4. *Consider* $d,d' \in \mathbb{Z}_{\geq 1}$, $\lambda \in \mathrm{T}_{\leq}(2,d)$, *and* $l = \lambda_2 - \lambda_1 + 1$. *For a set* U, *let* $f \in U$, $m \in U^d$, $m' \in U^{d'}$, *and* $m'' \in U^l$.

1. *Let* $m +\!\!\!+ m' = (m_1,\ldots,m_d,m'_1,\ldots,m'_{d'}) \in U^{d+d'}$ *for the concatenation. If* $\mu \in \mathbb{Z}_{\geq 1}$, *then let* $m^{+\!\!\!+\mu} = m +\!\!\!+ \cdots +\!\!\!+ m \in U^{\mu d}$ *for the* μ*-fold concatenation.*

2. *Let* $m^{\sqcup} = m +\!\!\!+ m^{\vee} \in U^{2d}$ *for the palindromization.*

3. *Let* $m[\lambda_1 : \lambda_2] = \left(m_{\lambda_1+i-1}\right)_{i \in [\![l]\!]} \in U^l$ *for the subtuple.*

4. *Suppose that* U *is a ring. Then, let*

$$m'' +_\lambda m = m +_\lambda m'' = m[1 : \lambda_1 - 1] +\!\!\!+ (m[\lambda_1 : \lambda_2] + m''[1 : l]) +\!\!\!+ m[\lambda_2 + 1 : d] \in U^d$$

for the segment addition. Also, let $m'' +_\lambda m = -(-m'' +_\lambda m)$ *and* $m -_\lambda m'' = -(m'' -_\lambda m)$ *for the segment subtraction.*

Notice that when $\mathrm{L}(m) = l$, then $m'' +_\lambda m = m +_\lambda m'' = m + m''$. Hence, $m'' \pm_\lambda m$ are well-defined.

2.2 Rings

We adopt the following notation for rings of polynomials, Laurent polynomials, and rational functions. Unless stated otherwise, let

$$\mathfrak{X} = \{X_1,\ldots,X_L\} = \{X_i\}_{i \in [\![L]\!]}$$

denote a finite set of free indeterminates $X_1,\ldots,X_L$ for some $L \in \mathbb{Z}_{\geq 1}$. We often refer to $\mathfrak{X}$ as a coordinate. If needed, we write $\mathfrak{X}_1 = \{X_{1,i}\}_{i \in [\![L_1]\!]}, \mathfrak{X}_2 = \{X_{2,i}\}_{i \in [\![L_2]\!]}, \ldots$ for multiple finite sets of free indeterminates.

First, $\mathbb{Q}[[\mathfrak{X}]] = \mathbb{Q}[[X_1,\ldots,X_L]]$, $\mathbb{Q}[\mathfrak{X}] = \mathbb{Q}[X_1,\ldots,X_L]$, and $\mathbb{Q}[\mathfrak{X}^{\pm 1}] = \mathbb{Q}[X_1^{\pm 1},\ldots,X_L^{\pm 1}]$ denote the rings of formal power series, polynomials, and Laurent polynomials, respectively. We write an element f in each of these rings as

$$f = \sum_{j \in \mathbb{Z}^L} f_{j_1,j_2,\ldots,j_L} X_1^{j_1} X_2^{j_2} \ldots X_L^{j_L}$$

possibly with zero $f_{j_1,j_2,\ldots,j_L} \in \mathbb{Q}$. If $0 \neq f \in \mathbb{Q}[\mathfrak{X}^{\pm 1}]$, then $\deg_{X_i} f \in \mathbb{Z}$ and $\mathrm{ord}_{X_i} f \in \mathbb{Z}$ denote the degree and order of f as the Laurent X_i-polynomial. Let $-\deg_{X_i}(0) = \mathrm{ord}_{X_i}(0) = \infty$ for our convention.

Second, $\mathbb{Q}(\mathfrak{X}) = \mathbb{Q}(X_1,\ldots,X_L)$ is the field of rational functions. We often look at the real values of rational functions. For this, let $f \in \mathbb{Q}(\mathfrak{X})$ and $r = (r_1,\ldots,r_L) \in \mathbb{R}^L$. Then, we assume

$$f(r) = f'(r) \in \mathbb{R},$$

if $f = f' \in \mathbb{Q}(\mathfrak{X})$ and $f'(r) \in \mathbb{R}$. We write $f(r)$ by $f(r_1,\ldots,r_L)$ as well. For a family $F = (F_i \in \mathbb{Q}(\mathfrak{X}))_{i \in I}$ and $r \in \mathbb{R}^L$, let $F(r) = (F(r)_i = F_i(r) \in \mathbb{R})_{i \in I}$.

2.3 Some q-analogs

We adopt the following tuple notation for the q-analogs in Definitions 1.4 and 1.15. For $l \in \mathbb{Z}_{\geq 1}$, we call $x \in \mathbb{Q}(\mathfrak{X})^l$ an indeterminate if each x_i is an indeterminate.

Definition 2.5. *Let $l \in \mathbb{Z}_{\geq 1}$ and $m, m', w \in \mathbb{Z}^l_{\geq 0}$. Suppose an indeterminate $x \in \mathbb{Q}(\mathfrak{X})^l$. Let*

$$\begin{aligned}
(m)^w_x &= \prod_{i \in [\![l]\!]} (m_i)^{w_i}_{x_i},\\
[m]^w_x &= \prod_{i \in [\![l]\!]} [m_i]^{w_i}_{x_i},\\
[m]!^w_x &= \prod_{i \in [\![l]\!]} [m_i]!^{w_i}_{x_i},\\
\begin{bmatrix} m \\ m' \end{bmatrix}^w_x &= \prod_{i \in [\![l]\!]} \begin{bmatrix} m_i \\ m'_i \end{bmatrix}^{w_i}_{x_i}
\end{aligned}$$

for the x-Pochhammer symbol, x-number, x-factorial, and x-binomial coefficient, respectively.

In particular, if $x = \iota^l(q)$ for an indeterminate $q \in \mathbb{Q}(\mathfrak{X})$, then let $(m)^w_q = (m)^w_x$, $[m]^w_q = [m]^w_x$, $[m]!^w_q = [m]!^w_x$, and $\left[\begin{smallmatrix} m \\ m' \end{smallmatrix}\right]^w_q = \left[\begin{smallmatrix} m \\ m' \end{smallmatrix}\right]^w_x$. Also, we often omit the superscript w when $w = \iota^l(1)$.

3 Fitting condition

We first introduce the following notions of gates, σ–operator, σ-plus and σ-minus, and σ-equivalence.

Definition 3.1. *Suppose $s \in \hat{\mathbb{Z}}^2$. We call s a gate if $s_1 < \infty$ and $s_1 \leq s_2$. Suppose that s is a gate.*

1. *We call $s_2 - s_1 + 1$ the width of s.*
2. *We call s finite if the width of s is finite; otherwise, we call s infinite.*

We now extend the notion of fitting condition in Definition 1.9.

Definition 3.2. *For $l \in \mathbb{Z}_{\geq 1}$, let $m \in \mathbb{Q}(\mathfrak{X})^l$ and $k, k' \in \mathbb{Q}(\mathfrak{X})^{2l}$.*

1. *We define the σ-operator $\sigma(k) \in \mathbb{Q}(\mathfrak{X})^l$ such that*

$$\sigma(k)_i = \sum k[i+1 : 2l - i + 1] \text{ for each } i \in [\![l]\!].$$

2. *In $\mathbb{Q}(\mathfrak{X})^l$, let*

$$\begin{aligned}
m \boxplus k &= m + \sigma(k)^\vee,\\
m \boxminus k &= m - \sigma(k).
\end{aligned}$$

We call $\boxplus$ and $\boxminus$ the σ-plus and σ-minus.

3. *We call k and k' σ-equivalent (or equivalent for short) if $\sigma(k) = \sigma(k')$.*

More explicitly, σ-equivalent k and k' satisfy the following equations:

$$\begin{aligned}
(m \boxplus k)_i &= m_i + \sigma(k)_{l-i+1} = (m \boxplus k')_i;\\
(m \boxminus k)_i &= m_i - \sigma(k)_i = (m \boxminus k')_i.
\end{aligned}$$

Remark 3.3. For $l \in \mathbb{Z}_{\geq 1}$, let $a \in \mathbb{Q}(\mathfrak{X})^l$. This allows expressions such as $a \mathbin{+\!\!+} a + a \mathbin{+\!\!+} a$ and $a \circ a - a$. To clarify, we adopt the following order of operations: (1) flip, palindromization; (2) $\circ$, scalar multiplication; (3) $\mathbin{+\!\!+}$, $\boxplus$, $\boxminus$; (4) tuple addition/subtraction; (5) scalar addition/subtraction. Other than the order of operations, we use the parentheses and center dots to avoid confusion.

Definition 3.4. *For $l \in \mathbb{Z}_{\geq 1}$, consider $m, n \in \mathbb{Z}^l$ and $k \in \mathbb{Z}^{2l}$ with a gate $s \geq 0$. Let $\mu = (s, l, m, n, k)$. We call l, m, n, and k the width, left ladder, right ladder, and support of μ. Also, we call k_1 the free parameter of μ.*

1. *In $\mathbb{Z}^{2l}$, we define*

$$\nu(k) = \left(\sum k[1:i]\right)_{i \in [\![2l]\!]},$$
$$\nu(m,n,k) = \nu(k) + m \mathbin{+\!\!+} n.$$

2. *Let $a = \nu(k)$ and $b = \nu(m,n,k)$. We call μ fitting if its ladders and support satisfy the following:*

$$m \mathbin{+\!\!+} n \in [\![s]\!]^{2l}, \tag{3.0.1}$$
$$b_1 \leq \ldots \leq b_l < b_{l+1} \leq \ldots \leq b_{2l}, \tag{3.0.2}$$
$$0 \leq a_1 \leq \ldots \leq a_l < a_{l+1} \leq \ldots \leq a_{2l}. \tag{3.0.3}$$

   We refer to (3.0.1)*,* (3.0.2)*, and* (3.0.3) *as the inclusion condition, upper slope condition, and lower slope condition of μ.*

3. *We call μ wrapped if $(m \boxminus k) \mathbin{+\!\!+} (n \boxplus k) \in [\![s]\!]^{2l}$.*

Compared to Definition 1.9, Definition 3.4 not only has the gate parameter s, but also employs the ladder and support parameters m, n, k for later computations.

Remark 3.5. Let $\mu = (s, l, m, n, k)$ be fitting. We call k_1 the free parameter of μ, since whenever $k_1 + \lambda \in \mathbb{Z}_{\geq 0}$, $\mu' = (s, l, m, n, (k_1 + \lambda) \mathbin{+\!\!+} k[2:l])$ is fitting. Also, μ is wrapped if and only if μ' is wrapped. Still, the free parameter gives different polynomials with positive integer coefficients later by the merged-log-concavity.

Example 3.6. Let $l = 2$. Suppose a fitting (s, l, m, n, k) with $a = \nu(k)$ and $b = \nu(m,n,k)$. Then, we have the following diagram, where m_i, n_i, and k_i indicate the differences along the inequalities.

$$\begin{array}{ccccc} & b_1 \leq & b_2 < & b_3 \leq & b_4 \\ & \mathbin{|\!\vee} m_1 & \mathbin{|\!\vee} m_2 & \mathbin{|\!\vee} n_1 & \mathbin{|\!\vee} n_2 \\ 0 \overset{k_1}{\leq} & a_1 \overset{k_2}{\leq} & a_2 \overset{k_3}{<} & a_3 \overset{k_4}{\leq} & a_4, \end{array}$$

Let us state the following lemmas for our later discussion.

Lemma 3.7. *Suppose a fitting (s, l, m, n, k) with $a = \nu(k)$ and $b = \nu(m,n,k)$. Then, we have the following inequalities.*

1. *$k = (a_1, a_2 - a_1, \cdots, a_{2l} - a_{2l-1}) \geq 0$.*
2. *$b \geq a \geq k_1 \geq 0$.*

Proof. Proof of Claim 1. By the lower slope condition, $k_1 = \sum k[1:1] = a_1 \geq 0$ and $k_i = \sum k[1:i] - \sum k[1:i-1] = a_i - a_{i-1} \geq 0$.

Proof of Claim 2. By the inclusion condition, $a \geq k_1$ by Claim 1 and $b - a = m \mathbin{+\!\!+} n \geq 0$. □

Lemma 3.8. *For $l \in \mathbb{Z}_{\geq 1}$, let $m, n \in \mathbb{Z}^l$ and $k \in \mathbb{Z}^{2l}$ with $a = \nu(k)$ and $b = \nu(m,n,k)$. Then, each $i \in [\![l]\!]$ gives $a_{2l-i+1} - a_i = \sigma(k)_i$ and $b_{2l-i+1} - b_i = n_{l-i+1} + \sigma(k)_i - m_i$.*

Proof. We have $a_{2l-i+1} - a_i = \sum k[1 : 2l-i+1] - \sum k[1 : i] = \sigma(k)_i$. Also, $b_{2l-i+1} - b_i = \sum k[1 : 2l-i+1] + n_{l-i+1} - (\sum k[1 : i] + m_i) = \sigma(k)_i + n_{l-i+1} - m_i$. □

Lemma 3.9. *Suppose a fitting (s,l,m,n,k). Then, we have the following inequalities:*

$$\sigma(k)_1 \geq \sigma(k)_2 \geq \ldots \geq \sigma(k)_l = k_{l+1} > 0;$$
$$n_l + \sigma(k)_1 - m_1 \geq n_{l-1} + \sigma(k)_2 - m_2 \geq \ldots \geq n_1 + \sigma(k)_l - m_l > 0.$$

Proof. Since $\sigma(k)_l = \sum k[l+1 : l+1] = k_{l+1}$, the inequalities hold by Lemma 3.8 and the slope conditions. □

In particular, we have the following equivalence for the width-one fitting condition.

Lemma 3.10. *For $l = 1$, suppose $m, n \in \mathbb{Z}^l$ and $k \in \mathbb{Z}^{2l}$ with a gate $s \geq 0$. Then, (s,l,m,n,k) is fitting if and only if $m \mathbin{+\!\!+} n \in [\![s]\!]^{2l}$, $k \geq (0,1)$, and $n_1 + k_2 > m_1$.*

Proof. First, $m \mathbin{+\!\!+} n \in [\![s]\!]^{2l}$ is the inclusion condition. Second, for $a = \nu(k)$, the lower slope condition is $a_1 = k_1 \geq 0$ and $a_2 - a_1 = k_2 \geq 1$. Third, for $b = \nu(m,n,k)$, the upper slope condition is $b_2 - b_1 = n_1 + k_2 - m_1 > 0$ by Lemma 3.8. □

4 Base shift functions

By gates and exponentiations, we extend the base shift function in Definition 1.7.

Definition 4.1. *Let $w \in \mathbb{Z}_{\geq 0}$, $\lambda \in \mathbb{Z}$, and $\rho \in \mathbb{Z}_{\geq 1}$ with a gate $s \geq 0$. Consider an indeterminate $q \in \mathbb{Q}(\mathfrak{X})$ and non-zero $\phi(q) \in \mathbb{Q}(q)$. Then, in $\mathbb{Q}(\mathfrak{X})$, we define the base shift function*

$$b(s,w,\lambda,\phi,\rho,q,\mathfrak{X}) = \begin{cases} b(\lambda,\phi,\rho,q)^w \text{ if } \lambda \in [\![s]\!], \\ 0 \text{ otherwise}. \end{cases}$$

We have the following positivities.

Lemma 4.2. *Let $\lambda \in [\![s]\!]$. Then, we have the following.*

1. *If either $w = 0$, $\lambda = 0$, or $\rho = 1$, then $b(s,w,\lambda,\phi,\rho,q,\mathfrak{X}) = 1$.*
2. *If $w = 1$ and $\lambda \geq 1$, then $b(s,w,\lambda,1-q,\rho,q,\mathfrak{X}) = \prod_{h \in [\![\lambda]\!]} [\rho]_{q^h} >_q 0$.*

Proof. Proof of Claim 1. When $\rho = 1$ or $\lambda = 0$, $\phi(q^\rho)^\lambda [\lambda]!_{q^\rho}$ and $\phi(q)^\lambda [\lambda]!_q$ of $b(s,w,\lambda,\phi,\rho,q,\mathfrak{X})$ coincide. Also, $w = 0$ implies $b(s,w,\lambda,\phi,\rho,q,\mathfrak{X}) = b(\lambda,\phi,\rho,q)^w = 1$ by $\lambda \in [\![s]\!]$.

Proof of Claim 2. We have $b(s,w,\lambda,1-q,\rho,q,\mathfrak{X}) = \frac{(\lambda)_{q^\rho}}{(\lambda)_q}$. □

Let us prove Lemma 1.8.

Proof. Statement 2 gives Statement 1 by Lemma 4.2. We prove the converse. If $\lambda = \rho = 2$, then $\frac{[\lambda]!_{q^\rho}}{[\lambda]!_q} = \frac{1+q^4}{1+q^2}$ is not a polynomial. This implies $\phi(q) \neq 1$. Suppose $\lambda = 1$. Then, $\frac{\phi(q^\rho)}{\phi(q)} >_q 0$ for each $\rho \in \mathbb{Z}_{\geq 1}$. Hence, $\phi(q)$ is a product of $\psi_m(q) = 1 - q^m$ for some $m \in \mathbb{Z}_{\geq 1}$, since $\phi(0) = 1$. Statement 2 now follows from the irreducibility of $\phi(q)$. □

For $l \in \mathbb{Z}_{\geq 1}$, $x \in \mathbb{Q}(\mathfrak{X})^l$, and $\phi(x) \in \prod_{i \in [\![l]\!]} \mathbb{Q}(x_i)$, we write $\phi(x)_i = \phi_i(x_i) \in \mathbb{Q}(x_i)$. We then define the following base shift functions over tuples.

Definition 4.3. *Assume a gate $s \geq 0$, $l \in \mathbb{Z}_{\geq 1}$, $w \in \mathbb{Z}^l_{\geq 0}$, $m,n \in \mathbb{Z}^l$, and $\rho \in \mathbb{Z}^l_{\geq 1}$. For an indeterminate $x \in \mathbb{Q}(\mathfrak{X})^l$, let $\phi(x) \in \prod_{i \in [\![l]\!]} \mathbb{Q}(x_i)$ such that $\phi_i(x_i) \neq 0$ for each $i \in [\![l]\!]$. The base shift functions are*

$$B(s,l,w,m,\phi,\rho,x,\mathfrak{X}) = \prod_{i \in [\![l]\!]} b(s,w_i,m_i,\phi_i,\rho_i,x_i,\mathfrak{X}) \in \mathbb{Q}(\mathfrak{X}),$$

$$B(s,l,w,m,n,\phi,\rho,x,\mathfrak{X}) = B(s,l,w,m,\phi,\rho,x,\mathfrak{X})B(s,l,w,n,\phi,\rho,x,\mathfrak{X}).$$

5 Squaring orders

We introduce the notion of squaring orders on rational functions. This is to discuss not only polynomials with positive integer coefficients, but also the positive real values of rational functions in some generality. We adopt the following notion (see [GilJer]).

Definition 5.1. *Suppose a set R.*

1. *A binary relation $\succeq$ on R is called a partial order if $\succeq$ satisfies the following conditions.*
   (a) *$f \succeq f$ for each $f \in R$ (reflexivity).*
   (b) *$f_1 \succeq f_2$ and $f_2 \succeq f_3$ imply $f_1 \succeq f_3$ (transitivity).*
   (c) *$f_1 \succeq f_2$ and $f_2 \succeq f_1$ imply $f_1 = f_2$ (antisymmetricity).*

   Suppose a binary relation $\succ$ on R. If $f \succ f$ never holds (irreflexivity) and $\succ$ has the transitivity, then $\succ$ is called a strict partial order on R. We also refer to a partial order and strict partial order as an inequality and strict inequality if no confusion occurs.

2. *Let R be a ring. Assume a partial order $\succeq$ on R. Then, R is called a partially ordered ring of $\succeq$ (or $\succeq$-poring for short) when R satisfies the following conditions.*
   (a) *$f_1 \succeq f_2$ and $f_3 \in R$ imply $f_1 + f_3 \succeq f_2 + f_3$ (additivity).*
   (b) *$f_1 \succeq 0$ and $f_2 \succeq 0$ imply $f_1 f_2 \succeq 0$ (multiplicativity).*

   Similarly, if a strict partial order $\succ$ on R satisfies the additivity and multiplicativity, then R is called a strictly partially ordered ring of $\succ$ (or strict $\succ$-poring).

Let us recall the following properties of porings.

Lemma 5.2. *If R is a $\succeq$-poring and strict $\succ$-poring, the following statements hold.*

1. (a) *$f \succeq g$ is equivalent to $f - g \succeq 0$.*
   (b) *$f_1 \succeq f_2$ and $g_1 \succeq g_2$ imply $f_1 + g_1 \succeq f_2 + g_2$.*
   (c) *$f_1 \succeq f_2$ and $g \succeq 0$ imply $f_1 g \succeq f_2 g$.*
   (d) *$f_1 \succeq f_2 \succeq 0$ and $g_1 \succeq g_2 \succeq 0$ imply $f_1 g_1 \succeq f_2 g_2 \succeq 0$.*
2. (a) *$f \succ g$ is equivalent to $f - g \succ 0$.*
   (b) *$f_1 \succ f_2$ and $g \succ 0$ imply $f_1 g \succ f_2 g$.*

(c) $f_1 \succ f_2$ *and* $g_1 \succ g_2$ *imply* $f_1 + g_1 \succ f_2 + g_2$.

(d) $f_1 \succ f_2 \succ 0$ *and* $g_1 \succ g_2 \succ 0$ *imply* $f_1 g_1 \succ f_2 g_2 \succ 0$.

Proof. Proof of Claim 1a. By the additivity and $-g \in R$, $f \succeq g$ implies $f - g \succeq g - g = 0$. Conversely, by $g \in R$, $f - g \succeq 0$ gives $f = f - g + g \succeq 0 + g = g$.

Proof of Claim 1b. Claim 1b holds by the transitivity of $\succeq$, since $f_1 + g_1 \succeq f_2 + g_1$ by $g_1 \in R$, and $f_2 + g_1 \succeq f_2 + g_2$ by $f_2 \in R$.

Proof of Claim 1c. Claim 1c follows from Claim 1a, because $f_1 - f_2 \succeq 0$ by $f_1 \succeq f_2$, and $(f_1 - f_2)g = f_1 g - f_2 g \succeq 0$ by the multiplicativity of $\succeq$.

Proof of Claim 1d. By Claim 1c, $f_1 \succeq f_2$ gives $f_1 g_1 \succeq f_2 g_1$. We also have $f_1 g_1 \succeq f_2 g_1 \succeq f_2 g_2$, since $g_1 \succeq g_2$ implies $f_2 g_1 \succeq f_2 g_2$. Hence, $f_1 g_1 \succeq f_2 g_2$ by the transitivity of $\succeq$, and $f_2 g_2 \succeq 0$ by the multiplicativity of $f_2, g_2 \succeq 0$.

We now obtain Claims 2a, 2c, 2b, and 2d, replacing $\succeq$ by $\succ$ in the above. □

Let us consider the following active domain of a binary relation.

Definition 5.3. *Suppose a binary relation* $\geq$ *on a set* R*. Then, let*

$$A(\geq, R) = \{ f \in R \mid f \geq g \text{ or } g \geq f \text{ for some } g \in R \}.$$

We now introduce the notion of squaring orders on general rings.

Definition 5.4. *Let* R *be a ring. Let* $\geq, >$ *and* $\succeq, \succ$ *be binary relations on* R*. Suppose the following conditions:*

1. $f > g$ *implies* $f \geq g$ *(*$>$*-*$\geq$ *implication);*
2. $f \succ g$ *implies* $f \succeq g$ *(*$\succ$*-*$\succeq$ *implication);*
3. $f \succeq 0$ *implies* $f \geq 0$ *(half* $\succeq$*-*$\geq$ *implication);*
4. $f \succ 0$ *implies* $f > 0$ *(half* $\succ$*-*$>$ *implication).*

We refer to these four implications as the squaring implications of $\geq, >$ *and* $\succeq, \succ$*.*

Also, suppose the following conditions:

1. *each of* $f \succeq g \succ h$ *and* $f \succ g \succeq h$ *implies* $f \succ h$ *(semi-strict transitivity);*
2. *we have the* $\geq$*-poring* $A(\geq, R)$ *and strict* $>$*-poring* $A(>, R)$ *such that* $A(\geq, R) = A(>, R)$ *(*$\geq$*-*$>$*-poring equality);*
3. *we have the* $\succeq$*-poring* $A(\succeq, R)$ *and strict* $\succ$*-poring* $A(\succ, R)$ *such that* $A(\succeq, R) = A(\succ, R)$ *(*$\succeq$*-*$\succ$*-poring equality);*
4. $A(\succeq, R) \subset A(\geq, R)$ *(*$\succeq$*-*$\geq$*-poring inclusion).*

Then, we call $\succeq, \succ$ *squaring orders on* $(R, \geq, >)$*. Also, we call* $\succ$ *a strict squaring order of* $\succeq$*.*

We employ the terminology "squaring orders" by the following implication diagram ("square diagram + ring" orders):

$$\begin{array}{ccccc} & & \succ & & \\ & \swarrow & & \searrow & \\ > & & & & \succeq \\ & \searrow & & \swarrow & \\ & & \geq & & \end{array}$$

Also, notice that $\succeq$ is not necessarily "larger than or equal to", since $f \succeq g$ and $f \neq g$ do not imply $f \succ g$.

Let us state the following lemma.

Lemma 5.5. *Suppose binary relations $\geq, >$ on a ring R with the $>$-$\geq$ implication and $>$-$\geq$-poring equality. Also, suppose binary relations $\succeq, \succ$ on R with the $\succeq$-$\geq$-poring inclusion and $\succeq$-$\succ$-poring equality.*

1. *The half $\succeq$-$\geq$ implication gives the $\succeq$-$\geq$ implication.*
2. *The half $\succ$-$>$ implication gives the $\succ$-$>$ implication.*

Proof. Proof of Claim 1. Let $f \succeq g$. Since $A(\succeq, R)$ is a $\succeq$-poring by the $\succeq$-$\succ$-poring equality, the additivity of $\succeq$ gives $f - g \succeq 0$. Then, $f - g \geq 0$ by the half $\succeq$-$\geq$ implication. Also, the $\succeq$-$\geq$-poring inclusion gives

$$f, g \in A(\succeq, R) \subset A(\geq, R).$$

We deduce $f \geq g$, since $A(\geq, R)$ is the $\geq$-poring by the $>$-$\geq$-poring equality.

Proof of Claim 2. Claim 2 follows similarly, since we have

$$A(\succ, R) = A(\succeq, R) \subset A(\geq, R) = A(>, R)$$

by the $\succeq$-$\succ$-poring equality, $\succeq$-$\geq$-poring inclusion, and $\geq$-$>$-poring equality. □

Then, we verify the following implications by squaring orders.

Proposition 5.6. *Squaring orders $\succeq, \succ$ on $(R, \geq, >)$ satisfy the following.*

1. *$f_1 \succeq f_2$ and $g_1 \succ g_2$ imply $f_1 + g_1 \succ f_2 + g_2$.*
2. *$f_1 \succ f_2 \succeq 0$ and $g_1 \succ g_2 \succeq 0$ imply $f_1 g_1 \succ f_2 g_2 \succeq 0$.*
3. *$f_1 \succeq f_2 \succ 0$ and $g_1 \succ g_2 \succeq 0$ imply $f_1 g_1 \succ f_2 g_2 \succeq 0$.*

Proof. Proof of Claim 1. By the $\succ$-$\succeq$ implication, $g_1 \in A(\succeq, R)$. Then,

$$f_1 + g_1 \succeq f_2 + g_1$$

by $f_1 \succeq f_2$ and the additivity of $\succeq$. Also, $f_2 \in A(\succ, R)$ by the $\succeq$-$\succ$-poring equality. We deduce

$$f_2 + g_1 \succ f_2 + g_2$$

by $g_1 \succ g_2$ and the additivity of $\succ$. Claim 1 holds by the semi-strict transitivity of $\succeq, \succ$.

Proof of Claim 2. By the semi-strict transitivity, $f_1 \succ f_2 \succeq 0$ implies $f_1 \succ 0$. Then, $f_1 g_1 \succ f_1 g_2$ follows from $g_1 \succ g_2$ and Claim 2b of Lemma 5.2. Furthermore, $f_1 \succeq f_2$ by the $\succ$-$\succeq$ implication. Then, $f_1 g_2 \succeq f_2 g_2$ by $g_2 \succeq 0$ and Claim 1c of Lemma 5.2. By the semi-strict transitivity, we deduce

$$f_1 g_1 \succ f_2 g_2.$$

Also, $f_2g_2 \succeq 0$ holds by $f_2, g_2 \succeq 0$ and the multiplicativity.

Proof of Claim 3. Since $g_1 \succ 0$ by the semi-strict transitivity on $g_1 \succ g_2 \succeq 0$, we have $g_1 \succeq 0$ by the $\succ$-$\succeq$ implication. Then, $f_1g_1 \succeq f_2g_1$ by $f_1 \succeq f_2$ and Claim 1c of Lemma 5.2. Also, since $g_1 \succ g_2$ and $f_2 \succ 0$, Claim 2b of Lemma 5.2 gives $f_2g_1 \succ f_2g_2$. Therefore,

$$f_1g_1 \succ f_2g_2$$

by the semi-strict transitivity. We also have $f_2g_2 \succeq 0$ by the multiplicativity, since $f_2 \succeq 0$ by the $\succ$-$\succeq$ implication on $f_2 \succ 0$, ☐

We state the following for our later discussion.

Corollary 5.7. *Suppose squaring orders $\succeq, \succ$ on $(R, \geq, >)$. Let $f_1, f_2, g_1, g_2 \in R$ such that $f_1 \succ f_2 \succeq 0$, $g_1 \succ 0$, $g_1 \succeq g_2$, and either $g_2 \succ 0$ or $g_2 = 0$. Then, we have $f_1g_1 \succ f_2g_2 \succeq 0$.*

Proof. The assumption says $g_1 \succeq g_2 \succ 0$ by $g_2 \succ 0$, or $g_1 \succ g_2 \succeq 0$ by $g_2 = 0$ and $g_1 \succ 0$. Hence, Claims 2 and 3 in Proposition 5.6 imply the assertion. ☐

Assume that a semiring $U \subset \mathbb{Q}$ satisfies $U \supsetneq \{0\}$, while $U \ni 1$ is not necessarily true. We call a semiring $U \subset \mathbb{Q}$ nonnegative if $U = U_{\geq 0} = \{u \in U \mid u \geq 0\}$. We introduce the following binary relations to obtain squaring orders.

Definition 5.8. *Let $\mathrm{O}_{\mathfrak{X}} = \left\{ r \in \mathbb{R}^L \mid 0 < r_i < 1 \text{ for each } i \in [\![L]\!] \right\}$. Consider a nonnegative semiring $U \subset \mathbb{Q}$. Then, we write the following six binary relations on $\mathbb{Q}(\mathfrak{X})$.*

1. *$f \geq^U_{\mathfrak{X}} g$ if $f, g \in \mathbb{Q}[\mathfrak{X}]$ and $f - g \in U[\mathfrak{X}]$. Also, $f >^U_{\mathfrak{X}} g$ if $f \geq^U_{\mathfrak{X}} g$ and $f \neq g$.*
2. *$f \geq^U_{\mathfrak{X}^{\pm 1}} g$ if $f, g \in \mathbb{Q}[\mathfrak{X}^{\pm 1}]$ and $f - g \in U[\mathfrak{X}^{\pm 1}]$. Also, $f >^U_{\mathfrak{X}^{\pm 1}} g$ if $f \geq^U_{\mathfrak{X}^{\pm 1}} g$ and $f \neq g$.*
3. *$f \geq_{\mathrm{O}_{\mathfrak{X}}} g$ if $f(r), g(r) \in \mathbb{R}$ and $f(r) \geq g(r)$ for each $r \in \mathrm{O}_{\mathfrak{X}}$. Also, $f >_{\mathrm{O}_{\mathfrak{X}}} g$ if $f \geq_{\mathrm{O}_{\mathfrak{X}}} g$ and $f(r) \neq g(r)$ for each $r \in \mathrm{O}_{\mathfrak{X}}$.*

For simplicity, let $\geq_{\mathfrak{X}} = \geq^{\mathbb{Z}_{\geq 0}}_{\mathfrak{X}}$, $>_{\mathfrak{X}} = >^{\mathbb{Z}_{\geq 0}}_{\mathfrak{X}}$, $\geq_{\mathfrak{X}^{\pm 1}} = \geq^{\mathbb{Z}_{\geq 0}}_{\mathfrak{X}^{\pm 1}}$, and $>_{\mathfrak{X}^{\pm 1}} = >^{\mathbb{Z}_{\geq 0}}_{\mathfrak{X}^{\pm 1}}$.

The open-unit hypercube $\mathrm{O}_{\mathfrak{X}}$ generalizes that of inequality (1.6.1). Also, $\mathrm{O}_{\mathfrak{X}}$ gives the $\geq_{\mathrm{O}_{\mathfrak{X}}}$- and strict $>_{\mathrm{O}_{\mathfrak{X}}}$-poring $A(\geq_{\mathrm{O}_{\mathfrak{X}}}, \mathbb{Q}(\mathfrak{X}))$. To prove this, we state the following lemmas.

Lemma 5.9. *Let $f \in \mathbb{Q}[\mathfrak{X}_1]$. Then, $f(r) = 0$ for each $r \in A_{\mathfrak{X}_1}$ if and only if $f = 0 \in \mathbb{Q}[\mathfrak{X}_1]$.*

Proof. The if part is clear. Let us prove the only if part. Suppose $\mathfrak{X}_1 = \{X_{1,i}\}_{i \in [\![L_1]\!]}$. When $L_1 = 1$, it follows from the division and the infinite cardinality of $A_{\mathfrak{X}_1}$. Let us use the induction on L_1. Suppose

$$\mathbb{Q}[\mathfrak{X}_1] \ni f = \sum_{j \in \mathbb{Z}^{L_1}} f_{j_1, \ldots, j_{L_1}} X^{j_1}_{1,1} \ldots X^{j_{L_1}}_{1,L_1} \neq 0.$$

This gives $\mu \in \mathbb{Z}^{L_1}$ such that $f_{\mu_1, \ldots, \mu_{L_1}} \neq 0$. Then, for $\mathfrak{X}_2 = \{X_{2,i} = X_{1,i+1}\}_{i \in [\![L_2]\!]}$ of $L_2 = L_1 - 1$, we have

$$\mathbb{Q}[\mathfrak{X}_2] \ni g = \sum_{j \in \mathbb{Z}^{L_2}} f_{\mu_1, j_1, \ldots, j_{L_2}} X^{j_1}_{2,1} \ldots X^{j_{L_2}}_{2,L_2} \neq 0.$$

The induction now gives $u \in A_{\mathfrak{X}_2}$ such that $g(u) \neq 0$. Also, $\mathbb{Q}[X_1] \ni h = f(X_1, u_1 \ldots, u_{L_2}) \neq 0$. Hence, the induction gives $v \in A_{\{X_1\}}$ such that $h(v) \neq 0$. This implies $w = (v_1, u_1, \ldots, u_{L_2}) \in \mathrm{O}_{\mathfrak{X}_1}$ such that $f(w) \neq 0$. ☐

Lemma 5.10. *If R is a strict $\succ$-poring such that $\emptyset \neq A(\succ, R) \subset R$, then $A(\succ, R) = R$.*

Proof. We have some $f \succ g$. Then, $f - g \succ 0$ by Claim 2a of Lemma 5.2. Hence, each $h \in R$ satisfies $f - g + h \succ h$ by the additivity of $\succ$. □

Furthermore, we adopt the following notation for our convenience.

Definition 5.11. *Suppose $F, G \in \mathbb{Q}(\mathfrak{X})^2$. Then in $\mathbb{Q}(\mathfrak{X})$, let*

$$\det(F,G) = \det \begin{bmatrix} F_1 & F_2 \\ G_1 & G_2 \end{bmatrix},$$

$$\mathrm{Frac}(F) = \frac{F_2}{F_1} \text{ if } F_1 \neq 0.$$

We obtain the following characterization and poring properties of $A(\geq_{\mathrm{O}_\mathfrak{X}}, \mathbb{Q}(\mathfrak{X}))$ and $A(>_{\mathrm{O}_\mathfrak{X}}, \mathbb{Q}(\mathfrak{X}))$.

Proposition 5.12. *We have the following.*

1. $A(\geq_{\mathrm{O}_\mathfrak{X}}, \mathbb{Q}(\mathfrak{X})) = \{f \in \mathbb{Q}(\mathfrak{X}) \mid f(r) \in \mathbb{R} \text{ for each } r \in \mathrm{O}_\mathfrak{X}\}$.
2. $A(\geq_{\mathrm{O}_\mathfrak{X}}, \mathbb{Q}(\mathfrak{X}))$ *is a* $\geq_{\mathrm{O}_\mathfrak{X}}$*-poring.*
3. $A(\geq_{\mathrm{O}_\mathfrak{X}}, \mathbb{Q}(\mathfrak{X}))$ *is a strict* $>_{\mathrm{O}_\mathfrak{X}}$*-poring.*
4. $A(\geq_{\mathrm{O}_\mathfrak{X}}, \mathbb{Q}(\mathfrak{X})) = A(>_{\mathrm{O}_\mathfrak{X}}, \mathbb{Q}(\mathfrak{X}))$.

Proof. Proof of Claim 1. Claim 1 holds by the reflexivity $f \geq_{\mathrm{O}_\mathfrak{X}} f$ for f in the right-hand side.

Proof of Claim 2. First, if $f_1 \geq_{\mathrm{O}_\mathfrak{X}} f_2 \geq_{\mathrm{O}_\mathfrak{X}} f_3$, then $f_1(r) - f_2(r) \geq 0$ and $f_2(r) - f_3(r) \geq 0$ for each $r \in \mathrm{O}_\mathfrak{X}$. Also, $f_1(r) - f_3(r) \geq 0$ for each $r \in \mathrm{O}_\mathfrak{X}$. Then, we obtain the transitivity $f_1 \geq_{\mathrm{O}_\mathfrak{X}} f_3$.

Second, if $f_1 \geq_{\mathrm{O}_\mathfrak{X}} f_2 \geq_{\mathrm{O}_\mathfrak{X}} f_1$, then $f_1(r) = f_2(r) \in \mathbb{R}$ for each $r \in \mathrm{O}_\mathfrak{X}$. This gives $F_1, F_2 \in \mathbb{Q}[\mathfrak{X}]^2$ such that $f_i = \mathrm{Frac}(F_i)$ with $\prod_{i \in [\![2]\!]} F_{i,1}(r) \neq 0$ and $\det(F_1, F_2)(r) = 0$ for each $r \in \mathrm{O}_\mathfrak{X}$. By Lemma 5.9, the antisymmetricity $f_1 = f_2$ holds.

Third, if $f_1 \geq_{\mathrm{O}_\mathfrak{X}} f_2$ and $f_3 \in A(\geq_{\mathrm{O}_\mathfrak{X}}, \mathbb{Q}(\mathfrak{X}))$, then the additivity $f_1 + f_3 \geq_{\mathrm{O}_\mathfrak{X}} f_2 + f_3$ holds by Claim 1 and $f_1(r) + f_3(r) - (f_2(r) + f_3(r)) = f_1(r) - f_2(r) \geq 0$ for each $r \in \mathrm{O}_\mathfrak{X}$.

Finally, if $f_1, f_2 \geq_{\mathrm{O}_\mathfrak{X}} 0$, then the multiplicativity $f_1 f_2 \geq_{\mathrm{O}_\mathfrak{X}} 0$ follows from $f_1(r) f_2(r) \geq 0$ for each $r \in \mathrm{O}_\mathfrak{X}$.

Proof of Claim 3. Claim 3 holds by the argument similar to the above without the reflexivity.

Proof of Claim 4. Claim 4 follows from Lemma 5.10, since $1 >_{\mathrm{O}_\mathfrak{X}} 0$. □

We now introduce the notions of squaring implications and squaring orders on $\mathfrak{X}$.

Definition 5.13. *We refer to the squaring implications of $\succeq, \succ$ and $\geq_{\mathrm{O}_\mathfrak{X}}, >_{\mathrm{O}_\mathfrak{X}}$ on $\mathbb{Q}(\mathfrak{X})$ as the squaring implications of $\succeq, \succ$ on $\mathfrak{X}$. Also, we refer to squaring orders $\succeq, \succ$ on $\left(\mathbb{Q}(\mathfrak{X}), \geq_{\mathrm{O}_\mathfrak{X}}, >_{\mathrm{O}_\mathfrak{X}}\right)$ as squaring orders on $\mathfrak{X}$.*

To obtain squaring orders on $\mathfrak{X}$, we state the following lemma.

Lemma 5.14. *We have the following.*

1. (a) $A(\geq^U_\mathfrak{X}, \mathbb{Q}(\mathfrak{X}))$ *is the* $\geq^U_\mathfrak{X}$*-poring such that* $A(\geq^U_\mathfrak{X}, \mathbb{Q}(\mathfrak{X})) = \mathbb{Q}[\mathfrak{X}]$.
   (b) $A(\geq^U_\mathfrak{X}, \mathbb{Q}(\mathfrak{X}))$ *is a strict* $>^U_\mathfrak{X}$*-poring.*
   (c) $A(\geq^U_\mathfrak{X}, \mathbb{Q}(\mathfrak{X})) = A(>^U_\mathfrak{X}, \mathbb{Q}(\mathfrak{X}))$.

2. (a) $A(\geq^U_{\mathfrak{X}^{\pm 1}}, \mathbb{Q}(\mathfrak{X}))$ *is the* $\geq^U_{\mathfrak{X}^{\pm 1}}$*-poring such that* $A(\geq^U_{\mathfrak{X}^{\pm 1}}, \mathbb{Q}(\mathfrak{X})) = \mathbb{Q}[\mathfrak{X}^{\pm 1}]$.

   (b) $A(\geq^U_{\mathfrak{X}^{\pm 1}}, \mathbb{Q}(\mathfrak{X}))$ *is a strict* $>^U_{\mathfrak{X}^{\pm 1}}$*-poring.*

   (c) $A(\geq^U_{\mathfrak{X}^{\pm 1}}, \mathbb{Q}(\mathfrak{X})) = A(>^U_{\mathfrak{X}^{\pm 1}}, \mathbb{Q}(\mathfrak{X}))$.

Proof. Proof of Claim 1a. First, by $0 \in U$, we have the reflexivity $f \geq^U_{\mathfrak{X}} f$ for $f \in A(\geq^U_{\mathfrak{X}}, \mathbb{Q}(\mathfrak{X}))$. Second, $A(\geq^U_{\mathfrak{X}}, \mathbb{Q}(\mathfrak{X})) = \mathbb{Q}[\mathfrak{X}]$ by the reflexivity. Third, let $f_1 \geq^U_{\mathfrak{X}} f_2 \geq^U_{\mathfrak{X}} f_3$. The transitivity $f_1 \geq^U_{\mathfrak{X}} f_3$ follows, since $f_1 - f_3 = (f_1 - f_2) + (f_2 - f_3) \in U[\mathfrak{X}]$ by the semiring U. Fourth, if $f_1 \geq^U_{\mathfrak{X}} f_2 \geq^U_{\mathfrak{X}} f_1$, then the antisymmetricity $f_1 = f_2$ follows, since $f_1 - f_2, f_2 - f_1 \in U[\mathfrak{X}]$ and $U = U_{\geq 0}$. Fifth, if $f_1 \geq^U_{\mathfrak{X}} f_2$ and $f_3 \in A(\geq^U_{\mathfrak{X}}, \mathbb{Q}(\mathfrak{X}))$, then the additivity $f_1 + f_3 \geq^U_{\mathfrak{X}} f_2 + f_3$ holds by $(f_1 + f_3) - (f_2 + f_3) = f_1 - f_3 \in U[\mathfrak{X}]$. Finally, $f_1, f_2 \geq^U_{\mathfrak{X}} 0$ implies the multiplicativity $f_1 f_2 \geq^U_{\mathfrak{X}} 0$ by the semiring U.

Proof of Claim 1b. Claim 1b follows as above, since $f >^U_{\mathfrak{X}} g$ demands $f \neq g$.

Proof of Claim 1c. Lemma 5.10 gives Claim 1c, because there is $f \in U$ such that $f >^U_{\mathfrak{X}} 0$ by $U \supsetneq \{0\}$.

Similar arguments hold for Claims 2a, 2b, and 2c. □

Hence, we have the following squaring orders.

Proposition 5.15. *The binary relations* $\geq^U_{\mathfrak{X}}$, $>^U_{\mathfrak{X}}$, $\geq^U_{\mathfrak{X}^{\pm 1}}$, $>^U_{\mathfrak{X}^{\pm 1}}$, $\geq_{\mathrm{O}_{\mathfrak{X}}}$, $>_{\mathrm{O}_{\mathfrak{X}}}$ *are squaring orders on* $\mathfrak{X}$ *such that* $>^U_{\mathfrak{X}}$, $>^U_{\mathfrak{X}^{\pm 1}}$, $>_{\mathrm{O}_{\mathfrak{X}}}$ *are strict squaring orders of* $\geq^U_{\mathfrak{X}}$, $\geq^U_{\mathfrak{X}^{\pm 1}}$, $\geq_{\mathrm{O}_{\mathfrak{X}}}$, *respectively.*

Proof. First, we establish that $\geq_{\mathrm{O}_{\mathfrak{X}}}, >_{\mathrm{O}_{\mathfrak{X}}}$ are squaring orders on $\mathfrak{X}$. By

Claims 2, 3, and 4 of Proposition 5.12, $\geq_{\mathrm{O}_{\mathfrak{X}}}, >_{\mathrm{O}_{\mathfrak{X}}}$ on $\mathfrak{X}$ satisfy the $\geq_{\mathrm{O}_{\mathfrak{X}}}$-$>_{\mathrm{O}_{\mathfrak{X}}}$-poring equality. Also, $\geq_{\mathrm{O}_{\mathfrak{X}}}, >_{\mathrm{O}_{\mathfrak{X}}}$ have the squaring implications on $\mathfrak{X}$ by Definition 5.8.

We prove the semi-strict transitivity of $\geq_{\mathrm{O}_{\mathfrak{X}}}, >_{\mathrm{O}_{\mathfrak{X}}}$. Assume

$$f_1 \geq_{\mathrm{O}_{\mathfrak{X}}} f_2 >_{\mathrm{O}_{\mathfrak{X}}} f_3.$$

Then, $f_1(r) - f_2(r) \geq 0$ and $f_2(r) - f_3(r) > 0$ for each $r \in \mathrm{O}_{\mathfrak{X}}$. We deduce $f_1 >_{\mathrm{O}_{\mathfrak{X}}} f_3$ by $f_1(r) - f_2(r) + f_2(r) - f_3(r) > 0$. Similarly, $f_1 >_{\mathrm{O}_{\mathfrak{X}}} f_2 \geq_{\mathrm{O}_{\mathfrak{X}}} f_3$ implies $f_1 >_{\mathrm{O}_{\mathfrak{X}}} f_3$. Therefore, $\geq_{\mathrm{O}_{\mathfrak{X}}}, >_{\mathrm{O}_{\mathfrak{X}}}$ are squaring orders on $\mathfrak{X}$ such that $>_{\mathrm{O}_{\mathfrak{X}}}$ is a strict squaring order of $\geq_{\mathrm{O}_{\mathfrak{X}}}$.

Second, we establish that $\geq^U_{\mathfrak{X}}, >^U_{\mathfrak{X}}$ are squaring orders on $\mathfrak{X}$. By Claim 1 of Lemma 5.14, $\geq^U_{\mathfrak{X}}, >^U_{\mathfrak{X}}$ give the $\geq_{\mathrm{O}_{\mathfrak{X}}}$-$>_{\mathrm{O}_{\mathfrak{X}}}$-poring equality. The squaring implications of $\geq^U_{\mathfrak{X}}$, $>^U_{\mathfrak{X}}$ on $\mathfrak{X}$ follows from Definition 5.8. Also, we have the $\geq^U_{\mathfrak{X}}$-$\geq_{\mathrm{O}_{\mathfrak{X}}}$-poring inclusion by Claim 1a, of Lemma 5.14 and Claims 1 and 2 of Proposition 5.12.

We prove the semi-strict transitivity of $\geq^U_{\mathfrak{X}}, >^U_{\mathfrak{X}}$. Suppose

$$f_1 >^U_{\mathfrak{X}} f_2 \geq^U_{\mathfrak{X}} f_3.$$

Then, $f_1 - f_2 \in U[\mathfrak{X}]$ with $f_1 - f_2 \neq 0$. Also, $f_2 - f_3 \in U[\mathfrak{X}]$. We obtain $f_1 >^U_{\mathfrak{X}} f_3$ by $f_1 - f_3 \in U[\mathfrak{X}]$, since $f_1 - f_3 = (f_1 - f_2) + (f_2 - f_3) \neq 0$ by $U = U_{\geq 0}$. Also, $f_1 \geq^U_{\mathfrak{X}} f_2 >^U_{\mathfrak{X}} f_3$ implies $f_1 >^U_{\mathfrak{X}} f_3$. Hence, $\geq^U_{\mathfrak{X}}, >^U_{\mathfrak{X}}$ are squaring orders such that $>^U_{\mathfrak{X}}$ is a strict squaring order of $\geq^U_{\mathfrak{X}}$.

Similar arguments hold for $\geq^U_{\mathfrak{X}^{\pm 1}}, >^U_{\mathfrak{X}^{\pm 1}}$. □

We compare squaring orders by the following terminology.

Definition 5.16. *Let* $\mathfrak{X}_1 \subset \mathbb{Q}(\mathfrak{X}_2)$. *Assume squaring orders* $O_i = \{\succeq_i, \succ_i\}$ *on* $\mathfrak{X}_i$ *for* $i \in [\![2]\!]$. *Then,* O_2 *is compatible to* O_1 *if the* $\succeq_1$-$\succeq_2$ *and* $\succ_1$-$\succ_2$ *implications hold. If* O_2 *is compatible to* O_1*, then we write*

$$O_2 \Supset O_1.$$

For instance, $\left\{\geq_{\mathrm{O}_{\mathfrak{X}}}, >_{\mathrm{O}_{\mathfrak{X}}}\right\} \Supset O$ for any squaring orders $O = \{\succeq, \succ\}$ on $\mathfrak{X}$ by Lemma 5.5.

5.1 Admissible variables

We discuss polynomials and their values in real numbers by squaring orders. For an indeterminate $x \in \mathbb{Q}(\mathfrak{X})$, the binary relation $>_x$ on $\mathbb{Q}(\mathfrak{X})$ does not have to be a squaring order on $\mathfrak{X}$. We define the following notion to obtain $>_x$ as a squaring order.

Definition 5.17. *Let $x \in \mathbb{Q}(\mathfrak{X})$ be an indeterminate. Suppose squaring orders $O = \{\succeq, \succ\}$ on $\mathfrak{X}$. We call x O-admissible (or admissible for short) if x and O satisfy the following conditions:*

1. *$f >_x 0$ implies $f \succ 0$ (half $>_x$-$\succ$ implication);*
2. *$1 >_{O_{\mathfrak{X}}} x$ (upper condition of x on $\mathfrak{X}$).*

Similarly, suppose an indeterminate $x \in \mathbb{Q}(\mathfrak{X})^l$ of $l \in \mathbb{Z}_{\geq 1}$. Then, we call x O-admissible (or admissible for short) if each x_i is O-admissible.

Also, we extend Definition 1.2 for an indeterminate $x \in \mathbb{Q}(\mathfrak{X})^l$, whose elements are not necessarily algebraically independent over $\mathbb{Q}$.

Definition 5.18. *For $l \in \mathbb{Z}_{\geq 1}$, suppose an indeterminate $x \in \mathbb{Q}(\mathfrak{X})^l$. Also, suppose a nonnegative semiring $U \subset \mathbb{Q}$.*

1. *Let $U[x] = \left\{ f \in \mathbb{Q}(\mathfrak{X}) \mid f = \sum_{j \in \mathbb{Z}^l_{\geq 0}} f_{j_1,\ldots,j_l} x_1^{j_1} \ldots x_l^{j_l} \text{for some finitely many non-zero } f_{j_1,\ldots,j_l} \in U \right\}$.*
2. *Let $U[x^{\pm 1}] = \left\{ f \in \mathbb{Q}(\mathfrak{X}) \mid f = \sum_{j \in \mathbb{Z}^l} f_{j_1,\ldots,j_l} x_1^{j_1} \ldots x_l^{j_l} \text{for some finitely many non-zero } f_{j_1,\ldots,j_l} \in U \right\}$.*
3. *Let $f \geq^U_x g$ if $f, g \in \mathbb{Q}[x]$ and $f - g \in U[x]$. Also, $f >^U_x g$ if $f \geq^U_x g$ and $f - g \neq 0$.*
4. *Let $f \geq^U_{x^{\pm 1}} g$ if $f, g \in \mathbb{Q}[x^{\pm 1}]$ and $f - g \in U[x^{\pm 1}]$. Also, $f >^U_{x^{\pm 1}} g$ if $f \geq^U_{x^{\pm 1}} g$ and $f - g \neq 0$.*

For simplicity, if $x = (q)$, then let $>^U_q = >^U_x$, $\geq^U_q = \geq^U_x$, $>^U_{q^{\pm 1}} = >^U_{x^{\pm 1}}$, and $\geq^U_{q^{\pm 1}} = \geq^U_{x^{\pm 1}}$. When $U = \mathbb{Z}_{\geq 0}$, we often omit the superscript ^{U} for the binary relations above.

By admissible variables, we shall obtain porings for these binary relations. First, we state the following general strict-to-non-strict transitivity by an indeterminate.

Lemma 5.19. *Suppose squaring orders $O = \{\succeq, \succ\}$ on $\mathfrak{X}$ and an indeterminate $x \in \mathbb{Q}(\mathfrak{X})$. Then, the half $>_x$-$\succ$ implication yields the half $\geq_x$-$\succeq$ implication.*

Proof. The assertion holds by the reflexivity of $\succeq$, since $f \geq_x 0$ implies $f >_x 0$ or $f = 0$. ☐

Second, we obtain the following inequality implications by admissible variables.

Lemma 5.20. *For squaring orders $O = \{\succeq, \succ\}$ on $\mathfrak{X}$, consider an O-admissible $x \in \mathbb{Q}(\mathfrak{X})^l$ of $l \in \mathbb{Z}_{\geq 1}$. Then, we have the following.*

1. *The half $\geq_x$-$\succeq$ implication holds.*
2. *The half $>_x$-$\succ$ implication holds.*
3. *$f \geq^U_x 0$ implies $\lambda \in \mathbb{Z}_{\geq 1}$ such that $\lambda f \succeq 0$ and $\lambda \mu f \succeq 0$ for each $\mu \geq_x 0$.*
4. *$f >^U_x 0$ implies $\lambda \in \mathbb{Z}_{\geq 1}$ such that $\lambda f \succ 0$ and $\lambda \mu f \succ 0$ for each $\mu >_x 0$.*

5. $f \geq_{x^{\pm 1}}^{U} 0$ *implies a monomial* $\lambda >_x 0$ *of variables* x_i *such that* $\lambda f \succeq 0$ *and* $\lambda \mu f \succeq 0$ *for each* $\mu \geq_x 0$.

6. $f >_{x^{\pm 1}}^{U} 0$ *implies a monomial* $\lambda >_x 0$ *of variables* x_i *such that* $\lambda f \succ 0$ *and* $\lambda \mu f \succ 0$ *for each* $\mu >_x 0$.

Proof. Proof of Claim 1. Since $f \geq_{x_i} 0$ implies $f \succeq 0$ by Lemma 5.19, Claim 1 follows from the additivity and multiplicativity of $\succeq$.

Proof of Claim 2. We deduce Claim 2 similarly from the half $>_{x_i}$-$\succ$ implication.

Proof of Claim 3. There exists $\lambda \in \mathbb{Z}_{\geq 1}$ such that $\lambda f \geq_x 0$ by $U = U_{\geq 0} \subset \mathbb{Q}$. It follows that $\lambda f \succeq 0$ by Claim 1. The latter inequality holds by the multiplicativity of $\succeq$, since $\mu \succeq 0$ by Claim 1.

Proof of Claim 4. We deduce Claim 4 similarly from Claim 2.

Proof of Claim 5. Recall that $f = \sum_{j \in \mathbb{Z}^l} f_{j_1,\dots,j_l} x_1^{j_1} \dots x_l^{j_l}$ for some finitely many $f_{j_1,\dots,j_l} \in \mathbb{Q}_{>0}$. Hence, there is $u \in \mathbb{Z}_{\geq 1}^l$ such that $u + j \geq 0$ whenever $f_{j_1,\dots,j_l} \in \mathbb{Q}_{>0}$. This gives a monomial $\lambda = v x_1^{u_1} \dots x_l^{u_l} >_x 0$ for some $v \in \mathbb{Z}_{\geq 1}$ such that $\lambda f \geq_x 0$. By Claim 1, we deduce $\lambda f \succeq 0$. The latter inequality follows from the multiplicativity of $\succeq$, since $\mu \succeq 0$ by Claim 1.

Proof of Claim 6. We have a monomial $\lambda = v x_1^{u_1} \dots x_l^{u_l} >_x 0$ for some $u \in \mathbb{Z}_{\geq 1}^l$ and $v \in \mathbb{Z}_{\geq 1}$ such that $\lambda f \geq_x 0$. Also, $\lambda f \neq 0$, since $\mathbb{Q}[\mathfrak{X}]$ is an integral domain and $\lambda \neq 0$. We obtain $\lambda f >_x 0$. In particular, $\lambda f \succ 0$ by Claim 2. The latter inequality holds by the multiplicativity of $\succ$ and Claim 2. □

By the inequality implications above, we deduce the following porings.

Proposition 5.21. *Suppose squaring orders $O = \{\succeq, \succ\}$ on $\mathfrak{X}$. Let $l \in \mathbb{Z}_{\geq 1}$ and $x \in \mathbb{Q}(\mathfrak{X})^l$ be O-admissible. Then, we have the following porings.*

1. (a) $A(\geq_x^U, \mathbb{Q}(\mathfrak{X}))$ *is the* $\geq_x^U$*-poring such that* $A(\geq_x^U, \mathbb{Q}(\mathfrak{X})) = \mathbb{Q}[x]$.
   (b) $A(\geq_x^U, \mathbb{Q}(\mathfrak{X}))$ *is a strict* $>_x^U$*-poring.*
   (c) $A(\geq_x^U, \mathbb{Q}(\mathfrak{X})) = A(>_x^U, \mathbb{Q}(\mathfrak{X}))$.
2. (a) $A(\geq_{x^{\pm 1}}^U, \mathbb{Q}(\mathfrak{X}))$ *is the* $\geq_{x^{\pm 1}}^U$*-poring such that* $A(\geq_{x^{\pm 1}}^U, \mathbb{Q}(\mathfrak{X})) = \mathbb{Q}[x^{\pm 1}]$.
   (b) $A(\geq_{x^{\pm 1}}^U, \mathbb{Q}(\mathfrak{X}))$ *is a strict* $>_{x^{\pm 1}}^U$*-poring.*
   (c) $A(\geq_{x^{\pm 1}}^U, \mathbb{Q}(\mathfrak{X})) = A(>_{x^{\pm 1}}^U, \mathbb{Q}(\mathfrak{X}))$.

Proof. Proof of Claim 1a. First, the reflexivity $f \geq_x^U f$ holds for $f \in A(\geq_x^U, \mathbb{Q}(\mathfrak{X}))$ by $0 \in U$. Also, $A(\geq_x^U, \mathbb{Q}(\mathfrak{X})) = \mathbb{Q}[x]$ by the reflexivity. Second, $f \geq_x^U g \geq_x^U h$ implies the transitivity $f \geq_x^U h$, since $f - g, g - h \in U[x]$ gives $(f-g)+(g-h) \in U[x]$ for the semiring U. Third, we prove the antisymmetricity. Let $f \geq_x^U g \geq_x^U f$. Then, Claim 3 of Lemma 5.20 gives $\lambda_1, \lambda_2 \in \mathbb{Z}_{\geq 1}$ such that $\lambda_1(f-g) \succeq 0$ and $\lambda_2(g-f) \succeq 0$. The former gives $\lambda_1\lambda_2(f-g) \succeq 0$ by $\lambda_2 \geq_x 0$ and Claim 3 of Lemma 5.20. Similarly, the latter gives $\lambda_1\lambda_2(g-f) \succeq 0$. Then, the antisymmetricity of $\succeq$ yields $\lambda_1\lambda_2 f = \lambda_1\lambda_2 g$. In particular, $\lambda_1\lambda_2(f-g) = 0$ and hence $f = g$, since $\mathbb{Q}[\mathfrak{X}]$ is an integral domain and $\lambda_1\lambda_2 \neq 0$. Furthermore, the additivity and multiplicativity of $\geq_x^U$ holds by the semiring U.

Proof of Claim 1b. The irreflexivity of $>_x^U$ follows, since for each $f \in A(\geq_x^U, \mathbb{Q}(\mathfrak{X}))$, we do not have $f >_x^U f$ by $f - f = 0$. Let us prove the transitivity. Consider $f >_x^U g >_x^U h$. This implies $f \geq_x^U h$ by the transitivity of $\geq_x^U$ and $>_x^U$-$\geq_x^U$ implication. Also, Claim 4 of Lemma 5.20 gives $\lambda \in \mathbb{Z}_{\geq 1}$ such that $\lambda(f-g), \lambda(g-h) \succ 0$. Then, $\lambda(f-g) + \lambda(g-h) = \lambda(f-h) \succ 0$ by the additivity of $\succ$. This gives $f - h \neq 0$ by the irreflexivity of $\succ$, and hence $f >_x^U h$. The additivity of $>_x^U$ on $A(\geq_x^U, \mathbb{Q}(\mathfrak{X}))$ holds by the semiring U. To verify the multiplicativity, let $f, g >_x^U 0$. Then, $fg \in U[x]$ by $f, g \in U[x]$. Also, $fg \neq 0$, since $\mathbb{Q}[\mathfrak{X}]$ is an integral domain. Hence, $fg >_x^U 0$ follows.

Proof of Claim 1c. Lemma 5.10 implies Claim 1c, since $U \neq \{0\}$ gives some $f, g \in U[x]$ such that $f >_x^U g$.

Claims 2a, 2b, and 2c hold similarly. □

Also, we state the following subset relations to obtain squaring orders by admissible variables.

Lemma 5.22. *Suppose squaring orders $O = \{\succeq, \succ\}$ on $\mathfrak{X}$. For $l \in \mathbb{Z}_{\geq 1}$, let $x \in \mathbb{Q}(\mathfrak{X})^l$ be O-admissible. Then, we have the following.*

1. *For each $r \in \mathrm{O}_{\mathfrak{X}}$ and $i \in [\![l]\!]$, we have $x_i(r) \in \mathbb{R}$ such that $0 < x_i(r) < 1$.*
2. *We have $A(\geq^U_x, \mathbb{Q}(\mathfrak{X})) \subset A(\geq_{\mathrm{O}_{\mathfrak{X}}}, \mathbb{Q}(\mathfrak{X}))$ and $A(\geq^U_{x^{\pm 1}}, \mathbb{Q}(\mathfrak{X})) \subset A(\geq_{\mathrm{O}_{\mathfrak{X}}}, \mathbb{Q}(\mathfrak{X}))$.*

Proof. Proof of Claim 1. We notice that $x_i \succ 0$ by $x_i >_{x_i} 0$ and the half $>_{x_i}$-$\succ$ implication. By the half $\succ$-$>_{\mathrm{O}_{\mathfrak{X}}}$ implication, we obtain $x_i >_{\mathrm{O}_{\mathfrak{X}}} 0$. This means $x_i(r) \in \mathbb{R}$ and $x_i(r) > 0$ for each $r \in \mathrm{O}_{\mathfrak{X}}$. We deduce Claim 1 from the upper condition of x_i.

Proof of Claim 2. Claim 2 holds by Claim 1. $\square$

We now obtain the following squaring orders by admissible variables, which are not necessarily algebraically independent over $\mathbb{Q}$.

Theorem 5.23. *Consider squaring orders $O = \{\succeq, \succ\}$ on $\mathfrak{X}$. For $l \in \mathbb{Z}_{\geq 1}$, let $x \in \mathbb{Q}(\mathfrak{X})^l$ be O-admissible. Then, $\geq^U_x, >^U_x, \geq^U_{x^{\pm 1}}, >^U_{x^{\pm 1}}$ are squaring orders on $\mathfrak{X}$ such that $>^U_x$ and $>^U_{x^{\pm 1}}$ are strict squaring orders of $\geq^U_x$ and $\geq^U_{x^{\pm 1}}$, respectively.*

Proof. We prove the assertion for $\geq^U_x, >^U_x$. The $>_{\mathrm{O}_{\mathfrak{X}}}$-$\geq_{\mathrm{O}_{\mathfrak{X}}}$ and $>^U_x$-$\geq^U_x$ implications are in Definitions 5.8 and 5.18. Claim 1 of Proposition 5.21 give the $\geq^U_x$-$>^U_x$-poring equality. The $\geq^U_x$-$\geq_{\mathrm{O}_{\mathfrak{X}}}$-poring inclusion holds by Claim 2 of Lemma 5.22.

First, we prove the half $\geq^U_x$-$\geq_{\mathrm{O}_{\mathfrak{X}}}$ implication. Let $f \geq^U_x 0$. We have $\lambda f \succeq 0$ for some $\lambda \in \mathbb{Z}_{\geq 1}$ by Claim 3 of Lemma 5.20. Also, $\lambda f \geq_{\mathrm{O}_{\mathfrak{X}}} 0$ by the half $\succeq$-$\geq_{\mathrm{O}_{\mathfrak{X}}}$ implication. By $\lambda \in \mathbb{Z}_{\geq 1}$, we deduce the half $\geq^U_x$-$\geq_{\mathrm{O}_{\mathfrak{X}}}$ implication $f \geq_{\mathrm{O}_{\mathfrak{X}}} 0$.

Second, the half $>^U_x$-$>_{\mathrm{O}_{\mathfrak{X}}}$ implication holds similarly by Claim 4 of Lemma 5.20 and the half $\succ$-$>_{\mathrm{O}_{\mathfrak{X}}}$ implication.

Third, we prove the semi-strict transitivity of $\geq^U_x, >^U_x$. Let $f \geq^U_x g >^U_x h$. Then, $f-g, g-h \in U[x]$ gives $f-h \in U[x]$, which implies $f \geq^U_x h$. Also, the $\geq^U_x$-$\geq_{\mathrm{O}_{\mathfrak{X}}}$ and $>^U_x$-$>_{\mathrm{O}_{\mathfrak{X}}}$ implications give $f \geq_{\mathrm{O}_{\mathfrak{X}}} g >_{\mathrm{O}_{\mathfrak{X}}} h$, and hence $f >_{\mathrm{O}_{\mathfrak{X}}} h$. Then, $f \neq h$ implies $f >^U_x h$. Similarly, we have $f >^U_x h$ by $f >^U_x g \geq^U_x h$.

The assertion for $\geq^U_{x^{\pm 1}}, >^U_{x^{\pm 1}}$ holds by a parallel argument. In particular, we replace Claims 3 and 4 of Lemma 5.20 with Claims 5 and 6 of Lemma 5.20 to prove the half $\geq^U_{x^{\pm 1}}$-$\geq_{\mathrm{O}_{\mathfrak{X}}}$ and $>^U_{x^{\pm 1}}$-$>_{\mathrm{O}_{\mathfrak{X}}}$ implications. $\square$

We have the following admissible variables.

Proposition 5.24. *Let $l \in \mathbb{Z}_{\geq 1}$. For squaring orders $O = \{\succeq, \succ\}$ on $\mathfrak{X}$, let $x \in \mathbb{Q}(\mathfrak{X})^l$ be O-admissible. Suppose an indeterminate $f = \sum_{j \in \mathbb{Z}^l_{\geq 0}} f_{j_1,\ldots,j_l} x_1^{j_1} \ldots x_l^{j_l} \in \mathbb{Q}_{\geq 0}[x]$.*

1. *f is O-admissible if f and O satisfy the following conditions:*
   - *(a) $f_{j_1,\ldots,j_l} \succ 0$ if $f_{j_1,\ldots,j_l} \neq 0$;*
   - *(b) $0 < \sum_{j \in \mathbb{Z}^l_{\geq 0}} f_{j_1,\ldots,j_l} \leq 1$.*
2. *f is O-admissible if $f = x_1^{j_1} \ldots x_l^{j_l}$ for some $j \in \mathbb{Z}^l_{\geq 0}$ such that $j \neq 0$.*

Proof. Proof of Claim 1. First, we prove

$$f \succ 0. \tag{5.1.1}$$

Since Claim 2 of Lemma 5.20 gives $x_1^{j_1} \dots x_l^{j_l} \succ 0$ for each $j \in \mathbb{Z}_{\geq 0}^l$, $f_{j_1,\dots,j_l} \neq 0$ implies $f_{j_1,\dots,j_l} x_1^{j_1} \dots x_l^{j_l} \succ 0$ by Condition 1a and the multiplicativity of $\succ$. We deduce inequality (5.1.1) by the additivity of $\succ$.

Second, we prove the half $>_f$-$\succ$ implication. For each $u \in \mathbb{Z}_{\geq 1}$, we have $u \succ 0$ by Claim 2 of Lemma 5.20 and the multiplicativity of $\succ$. Also, for each $d \in \mathbb{Z}_{\geq 0}$, $f^d \succ 0$ by inequality (5.1.1) and the multiplicativity of $\succ$. Consequently, if $g >_f 0$, then $g \succ 0$ by the additivity and multiplicativity of $\succ$.

Third, the upper condition of f on $\mathfrak{X}$ holds as follows. Since f is an indeterminate, there is $j \in \mathbb{Z}_{\geq 0}^l$ such that $f_{j_1,\cdots,j_l} x_1^{j_1} \dots x_l^{j_l} \notin \mathbb{Q}$. It follows that $1 >_{\mathrm{O}_{\mathfrak{X}}} f$ by Condition 1b, since Claim 1 of Lemma 5.22 implies $0 < x_i(r) < 1$ for $i \in [\![l]\!]$ and $r \in \mathrm{O}_{\mathfrak{X}}$.

Proof of Claim 2. Claim 1 and $1 \succ 0$ imply Claim 2. □

By admissible variables, we define the following notion to discuss explicit real values of rational functions over $\mathrm{O}_{\mathfrak{X}}$.

Definition 5.25. *Suppose squaring orders $O = \{\succeq, \succ\}$ on $\mathfrak{X}$. Let $l \in \mathbb{Z}_{\geq 1}$.*

1. *We call $x \in \mathbb{Q}(\mathfrak{X})$ fully O-admissible by $\mathfrak{X}$ (or fully admissible for short) if x is O-admissible and there exist $d \in \mathbb{Z}_{\geq 1}$ and $X_i \in \mathfrak{X}$ such that $x = X_i^d$.*
2. *We call $x \in \mathbb{Q}(\mathfrak{X})^l$ fully O-admissible by $\mathfrak{X}$ (or fully admissible for short) if each $x_i \in \mathbb{Q}(\mathfrak{X})$ is fully O-admissible by $\mathfrak{X}$.*

5.2 Mediators

We introduce the notion of mediators, extending the q-Pochhammer symbols $(n)_q$ within our purpose. Since a strict $\succ$ presumes a non-strict $\succeq$, we often denote squaring orders $\{\succeq, \succ\}$ just by $\succ$.

Definition 5.26. *Consider a gate $s \geq 0$, $l \in \mathbb{Z}_{\geq 1}$, $w \in \mathbb{Z}_{\geq 0}^l$, and $\rho \in \mathbb{Z}_{\geq 1}^l$. Suppose a $\succ$-admissible $x \in \mathbb{Q}(\mathfrak{X})^l$ and $\phi(x) \in \prod_{i \in [\![l]\!]} \mathbb{Q}(x_i)$. Let $\mu = (s, l, w, \succ, \rho, x, \mathfrak{X})$. We call ϕ a μ-mediator (or a mediator for short) if ϕ and μ satisfy the following conditions.*

1. *For each $i \in [\![l]\!]$, $\phi(x)_i^{w_i} >_{\mathrm{O}_{\mathfrak{X}}} 0$.*
2. *For each $m \in [\![s]\!]^l$, $B(s, l, w, m, \phi, \rho, x, \mathfrak{X}) \succ 0$.*

We refer to 1 and 2 as the base positivity and base-shift positivity of ϕ and μ.

For a $\succ$-admissible variable $q \in \mathbb{Q}(\mathfrak{X})$, we have the squaring order $>_q$ of q-polynomials by Theorem 5.23. Then, we introduce the notion of canonical mediators by Lemma 1.8.

Definition 5.27. *Let $l \in \mathbb{Z}_{\geq 1}$. If $\phi(x) = (1 - x_i)_{i \in [\![l]\!]} \in \prod_{i \in [\![l]\!]} \mathbb{Q}(x_i)$ for an indeterminate $x \in \mathbb{Q}(\mathfrak{X})^l$, then we call ϕ the canonical l-mediator (or the canonical mediator for short).*

When $l = 1$ and $x = (q) \in \mathbb{Q}(\mathfrak{X})^l$, the canonical mediator $\phi(x) = (1 - q) \in \mathbb{Q}(\mathfrak{X})^l$ gives the q-Pochhammer symbols $\phi(x)_1[n]_q = (n)_q$ of $n \in \mathbb{Z}_{\geq 0}$.

6 Merged-log-concavity

We now introduce the notions of ring shift factors, merged determinants, parcels, and merged-log-concavity.

Definition 6.1. *Suppose a gate $s \geq 0$. Let $l \in \mathbb{Z}_{\geq 1}$, $w \in \mathbb{Z}^l_{\geq 0}$, and $\rho \in \mathbb{Z}^l_{\geq 1}$. Assume squaring orders $O = \{\succeq, \succ\}$ on $\mathfrak{X}$. Consider a $(s,l,w,\succ,\rho,x,\mathfrak{X})$-mediator ϕ for an O-admissible $x \in \mathbb{Q}(\mathfrak{X})^l$.*

1. *If $\mathscr{F} = (\mathscr{F}_m \in \mathbb{Q}(\mathfrak{X}))_{m\in\mathbb{Z}^l}$, then for $m,n \in \mathbb{Z}^l$ and $k \in \mathbb{Z}^{2l}$, let*

$$\det(\mathscr{F},m,n,k) = \det\begin{bmatrix} \mathscr{F}_m & \mathscr{F}_{(n\boxplus k)^\vee} \\ \mathscr{F}_{m\boxminus k} & \mathscr{F}_{n^\vee}\end{bmatrix}.$$

2. *Let $m,n \in \mathbb{Z}^l$ and $k \in \mathbb{Z}^{2l}$ with $a = \nu(k)$ and $b = \nu(m,n,k)$. Suppose $y = x^\rho$. Then, in $\mathbb{Q}(\mathfrak{X})$, we define the ring shift factor*

$$\Upsilon(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) = \begin{cases} \prod(\phi(y)^{\sqcup})^{(b-a)\circ w^{\sqcup}} \cdot \dfrac{[b]!^{w^{\sqcup}}_{y^{\sqcup}}}{[a]!^{w^{\sqcup}}_{y^{\sqcup}}} \text{ if } a,b \geq 0, \\ 0 \text{ otherwise}. \end{cases}$$

   Also, we define the merged determinant

$$\Delta(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) = \Upsilon(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X})\det(\mathscr{F},m,n,k) \in \mathbb{Q}(\mathfrak{X}).$$

3. *When $f_s = (f_{s,m} \in \mathbb{Q}(\mathfrak{X}))_{m\in\mathbb{Z}^l}$ satisfies*

$$\begin{cases} f_{s,m} \succ 0 \text{ if } m \in [\![s]\!]^l, \\ f_{s,m} = 0 \text{ otherwise}, \end{cases}$$

   we call f_s $(s,l,\succ)$-positive (or $\succ$-positive for short).

4. *Suppose a $\succ$-positive $f_s = (f_{s,m} \in \mathbb{Q}(\mathfrak{X}))_{m\in\mathbb{Z}^l}$. Then, we define the parcel*

$$\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X}) = (\mathscr{F}_m \in \mathbb{Q}(\mathfrak{X}))_{m\in\mathbb{Z}^l}$$

   by the following rational functions:

$$\mathscr{F}_m = \begin{cases} \dfrac{f_{s,m}}{\prod \phi(x)^{m\circ w} \cdot [m]!^w_x} \text{ if } m \in [\![s]\!]^l, \\ 0 \text{ otherwise}. \end{cases}$$

   We refer to s, l, w, $\succ$, f_s, ϕ, ρ, x, and $\mathfrak{X}$ as the gate, width, weight, strict squaring order, numerator, mediator, base shift, base, and coordinate of $\mathscr{F}$. We call them parcel parameters of $\mathscr{F}$.

5. *Suppose $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ with squaring orders $O' = \{\succeq', \succ'\} \supseteq O$.*

   (a) *We call $\mathscr{F}$ $(s,l,w,\succ',\phi,\rho,x,\mathfrak{X})$-merged-log-concave (or $\succ'$-merged-log-concave for short) if any fitting (s,l,m,n,k) satisfies*

$$\Delta(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) \succ' 0.$$

(b) *Similarly, we call* $\mathscr{F}$ $(s,l,w,\succeq',\phi,\rho,x,\mathfrak{X})$*-merged-log-concave (or* $\succeq'$*-merged-log-concave for short),if any fitting* (s,l,m,n,k) *satisfies*

$$\Delta(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) \succeq' 0.$$

Suppose width-l parcels $\mathscr{F}$ and $\mathscr{F}'$. As families of rational functions, we consider $\mathscr{F} = \mathscr{F}'$ if $\mathscr{F}_m = \mathscr{F}'_m \in \mathbb{Q}(\mathfrak{X})$ for each $m \in \mathbb{Z}^l$ even with different parcel parameters. We later discuss the change of parcel parameters and the merged-log-concavity in Propositions 7.8 and 18.1.

We simplify some notations in Definition 6.1 for the following cases.

Definition 6.2. *Suppose a parcel* $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$.

1. *When* ϕ *is the canonical mediator, we write*

$$\begin{aligned}\Upsilon(s,l,w,m,n,k,\rho,x,\mathfrak{X}) &= \Upsilon(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}),\\ \Delta(\mathscr{F})(s,l,w,m,n,k,\rho,x,\mathfrak{X}) &= \Delta(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}),\\ \Lambda(s,l,w,\succ,f_s,\rho,x,\mathfrak{X}) &= \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X}).\end{aligned}$$

2. *When* $\rho = \iota^l(1)$*, we write*

$$\begin{aligned}\Upsilon(s,l,w,m,n,k,\phi,x,\mathfrak{X}) &= \Upsilon(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}),\\ \Delta(\mathscr{F})(s,l,w,m,n,k,\phi,x,\mathfrak{X}) &= \Delta(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}),\\ \Lambda(s,l,w,\succ,f_s,\phi,x,\mathfrak{X}) &= \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X}).\end{aligned}$$

3. *When* $\rho = \iota^l(1)$ *and* ϕ *is the canonical mediator, we write*

$$\begin{aligned}\Upsilon(s,l,w,m,n,k,x,\mathfrak{X}) &= \Upsilon(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}),\\ \Delta(\mathscr{F})(s,l,w,m,n,k,x,\mathfrak{X}) &= \Delta(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}),\\ \Lambda(s,l,w,\succ,f_s,x,\mathfrak{X}) &= \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X}).\end{aligned}$$

4. *When* $w = \iota^l(0)$*, we write*

$$\begin{aligned}\Upsilon(s,l,m,n,k,\mathfrak{X}) &= \Upsilon(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}),\\ \Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X}) &= \Delta(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}),\\ \Lambda(s,l,\succ,f_s,\mathfrak{X}) &= \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X}).\end{aligned}$$

7 Fundamental discussions on parcels and the merged-log-concavity

7.1 On the mediators

On the canonical mediators, we obtain the following q-Pochhammer symbols.

Proposition 7.1. *Let* $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\rho,x,\mathfrak{X})$.

1. *Then, we have*

$$\mathscr{F}_m = \begin{cases} \dfrac{f_{s,m}}{(m)_x^w} \text{ for } m \in [\![s]\!]^l, \\ 0 \text{ otherwise}. \end{cases} \tag{7.1.1}$$

2. *Let $m,n \in \mathbb{Z}^l$, $k \in \mathbb{Z}^{2l}$, $\rho \in \mathbb{Z}^l_{\geq 1}$, and $y = x^\rho$ with $a = \nu(k)$ and $b = \nu(m,n,k)$. Then, we have*

$$\Delta(\mathscr{F})(s,l,w,m,n,k,\rho,x,\mathfrak{X}) = \begin{cases} \frac{(b)^{w^\sqcup}_{y^\sqcup}}{(a)^{w^\sqcup}_{y^\sqcup}} \cdot \det(\mathscr{F},m,n,k) \text{ if } a,b \geq 0, \\ 0 \text{ otherwise.} \end{cases} \tag{7.1.2}$$

Proof. *Proof of Claim* (7.1.1). We have $m \geq 0$ by $m \in [\![s]\!]^l$. Then, Claim (7.1.1) holds by

$$\prod \phi(x)^{m \circ w} \cdot [m]!^w_x = \prod_{i \in [\![l]\!]} (1-x_i)^{m_i w_i} [m_i]!^{w_i}_{x_i} = (m)^w_x.$$

Proof of Claim (7.1.2). We obtain Claim (7.1.2), since $a,b \geq 0$ gives

$$(\phi(y)^\sqcup)^{b \circ w^\sqcup} \cdot [b]!^{w^\sqcup}_{y^\sqcup} = \prod_{i \in [\![l]\!]} (1-y_i)^{b_i w_i} [b_i]^{w_i}_{y_i} \cdot \prod_{i \in [\![l]\!]} (1-y_{l-i+1})^{b_{i+l} w_{l-i+1}} [b_{i+l}]^{w_{l-i+1}}_{y_{l-i+1}} = \prod_{i \in [\![l]\!]} (b_i)^{w_i}_{y_i} (b_{i+l})^{w_{l-i+1}}_{y_{l-i+1}} = (b)^{w^\sqcup}_{y^\sqcup}$$

and similarly $(\phi(y)^\sqcup)^{a \circ w^\sqcup} \cdot [a]!^{w^\sqcup}_{y^\sqcup} = (a)^{w^\sqcup}_{y^\sqcup}$. □

On the choice of mediators, we obtain the following invariance of the merged-log-concavity for the trivial base shifts.

Proposition 7.2. *Consider $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,x,\mathfrak{X})$ and $\mathscr{G} = \Lambda(s,l,w,\succ,f_s,\psi,x,\mathfrak{X})$. For a fitting $\mu = (s,l,m,n,k)$, let $a = \nu(k)$ and $b = \nu(m,n,k)$. Then, we have*

$$\frac{\prod(\phi(x)^\sqcup)^{(b-a)\circ w^\sqcup}}{\prod \phi(x)^{m \circ w} \cdot \prod \phi(x)^{n^\vee \circ w}} = \frac{\prod(\psi(x)^\sqcup)^{(b-a)\circ w^\sqcup}}{\prod \psi(x)^{m \circ w} \cdot \prod \psi(x)^{n^\vee \circ w}} = 1, \tag{7.1.3}$$

$$\frac{\prod(\phi(x)^\sqcup)^{(b-a)\circ w^\sqcup}}{\prod \phi(x)^{(m \boxminus k)\circ w} \cdot \prod \phi(x)^{(n \boxplus k)^\vee \circ w}} = \frac{\prod(\psi(x)^\sqcup)^{(b-a)\circ w^\sqcup}}{\prod \psi(x)^{(m \boxminus k)\circ w} \cdot \prod \psi(x)^{(n \boxplus k)^\vee \circ w}} = 1. \tag{7.1.4}$$

In particular, $\mathscr{F}$ is $\succ'$-merged-log-concave if and only if $\mathscr{G}$ is $\succ'$-merged-log-concave. Similarly, $\mathscr{F}$ is $\succeq'$-merged-log-concave if and only if $\mathscr{G}$ is $\succeq'$-merged-log-concave.

Proof. First, we prove equation (7.1.3). By

$$(\phi(x)^{n^\vee \circ w})^\vee = (\phi(x)^\vee)^{(n^\vee \circ w)^\vee} = (\phi(x)^\vee)^{n \circ w^\vee},$$

we have $\prod \phi(x)^{n^\vee \circ w} = \prod(\phi(x)^\vee)^{n \circ w^\vee}$. This gives $\prod \phi(x)^{m \circ w} \cdot \prod \phi(x)^{n^\vee \circ w} = \prod(\phi(x) \mathbin{+\!\!\!+} \phi(x)^\vee)^{(m \mathbin{+\!\!\!+} n)\circ(w \mathbin{+\!\!\!+} w^\vee)} = \prod(\phi(x)^\sqcup)^{(m \mathbin{+\!\!\!+} n)\circ w^\sqcup}$. By $m \mathbin{+\!\!\!+} n = b - a$, we deduce equation (7.1.3).

Second, equation (7.1.4) follows from equation (7.1.3), because

$$\prod \phi(x)^{(m \boxminus k)\circ w} \cdot \prod \phi(x)^{(n \boxplus k)^\vee \circ w} = \prod_{i \in [\![l]\!]} \phi(x)_i^{(m_i - \sigma(k)_i) w_i} \cdot \phi(x)_i^{(n_{l-i+1} + \sigma(k)_i) w_i} = \prod \phi(x)^{m \circ w} \cdot \prod \phi(x)^{n^\vee \circ w}.$$

Third, latter statements hold by equations (7.1.3) and (7.1.4), since when μ is fitting,

$$\Upsilon(s,l,w,m,n,k,\phi,x,\mathfrak{X}) \mathscr{F}_m \mathscr{F}_{n^\vee} = \frac{\prod(\phi(x)^\sqcup)^{(b-a)\circ w^\sqcup}}{\prod \phi(x)^{m \circ w} \cdot \prod \phi(x)^{n^\vee \circ w}} \frac{[b]!^{w^\sqcup}_{x^\sqcup}}{[a]!^{w^\sqcup}_{x^\sqcup}} \frac{f_m f_{n^\vee}}{[m]!^w_x [n^\vee]!^w_x},$$

and when μ is wrapped and fitting,

$$\Upsilon(s,l,w,m,n,k,\phi,x,\mathfrak{X})\mathscr{F}_{m\boxminus k}\mathscr{F}_{(n\boxplus k)^\vee} = \frac{\prod(\phi(x)^{\sqcup})^{(b-a)\circ w^{\sqcup}}}{\prod\phi(x)^{(m\boxminus k)\circ w}\cdot\prod\phi(x)^{(n\boxplus k)^\vee\circ w}}\frac{[b]!^{w^{\sqcup}}_{x^{\sqcup}}}{[a]!^{w^{\sqcup}}_{x^{\sqcup}}}\frac{f_{m\boxminus k}f_{(n\boxplus k)^\vee}}{[m\boxminus k]!^{w}_{x}[(n\boxplus k)^\vee]!^{w}_{x}}.$$

□

In Proposition 7.2. $\Delta(\mathscr{F})(s,l,w,m,n,k,\phi,x,\mathfrak{X}) = \Delta(\mathscr{G})(s,l,w,m,n,k,\psi,x,\mathfrak{X})$. But, the choice of mediators matters for explicit $\mathscr{F}_m(r),\mathscr{G}_m(r)\in\mathbb{R}$, since $r\in \mathrm{O}_{\mathfrak{X}}$ does not satisfy $\mathscr{F}_m(r)=\mathscr{G}_m(r)\in\mathbb{R}$ in general.

7.2 On the coordinates

We introduce the following notion on the choice of parcel coordinates by $\iota^\lambda(1)$ in Definition 1.6.

Definition 7.3. *Let $\mathfrak{X}_1=\{X_{1,i}\}_{i\in[\![L_1]\!]}$.*

1. *If $\lambda\subset[\![L_1]\!]$, then for a family $\kappa\in\prod_{i\in\lambda}\mathbb{Z}_{\geq 1}$, we define the set*

$$r_{\lambda,\kappa}(\mathfrak{X}_1)=\left\{X_i^{\kappa_i}\right\}_{i\in\lambda},$$

   which we call a restricted coordinate of $\mathfrak{X}_1$.

2. *In particular, if $\kappa=\iota^\lambda(1)$, then let*

$$r_\lambda(\mathfrak{X}_1)=r_{\lambda,\kappa}(\mathfrak{X}_1).$$

Suppose $\mathscr{F}=\Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X}_1)$ and a restricted coordinate $\mathfrak{X}_2=r_{\lambda,\kappa}(\mathfrak{X}_1)$. Let $\mu=(\mathscr{F},\mathfrak{X}_2)$.

(a) *We say the following:*

- *μ satisfies the base condition if $x\in\mathbb{Q}(\mathfrak{X}_2)^l$;*
- *μ satisfies the numerator condition if $f_{s,m}\in\mathbb{Q}(\mathfrak{X}_2)$ for each $m\in\mathbb{Z}^l$.*

(b) *When the base and numerator conditions of μ hold if and only if $\kappa=\iota^\lambda(1)$, we call $\mathfrak{X}_2$ optimal for $\mathscr{F}$.*

By the notion above, we consider the change of coordinates $\mathfrak{X}_2\subset\mathbb{Q}(\mathfrak{X}_1)$ such that $\mathfrak{X}_2$ has powers of some elements of $\mathfrak{X}_2$, and $\mathscr{F}$ is still a parcel on $\mathbb{Q}(\mathfrak{X}_2)$. Then, we demonstrate the existence of optimal coordinates for arbitrary parcels. For this, we introduce the following notion of faithful squaring orders.

Definition 7.4. *Assume squaring orders $O_1=\{\succeq_1,\succ_1\}$ on $\mathfrak{X}_1$. We call O_1 faithful if for any $\mathfrak{X}_2=r_{\lambda,\kappa}(\mathfrak{X}_1)$, the binary relations $\succeq_2=\succeq|_{\mathbb{Q}(\mathfrak{X}_2)}$ and $\succ_2=\succ|_{\mathbb{Q}(\mathfrak{X}_2)}$ give the $\succeq_2$-$\succ_2$-poring equality*

$$A(\succeq_2,\mathbb{Q}(\mathfrak{X}_2))=A(\succ_2,\mathbb{Q}(\mathfrak{X}_2)).$$

We obtain faithful squaring orders.

Lemma 7.5. *Assume squaring orders $O_1=\{\succeq_1,\succ_1\}$ on $\mathfrak{X}_1$. For $\mathfrak{X}_2=r_{\lambda,\kappa}(\mathfrak{X}_1)$, let $\succeq_2=\succeq_1|_{\mathbb{Q}(\mathfrak{X}_2)}$ and $\succ_2=\succ_1|_{\mathbb{Q}(\mathfrak{X}_2)}$. Then, we have the following.*

1. *$A(\succeq_2,\mathbb{Q}(\mathfrak{X}_2))=A(\succeq_1,\mathbb{Q}(\mathfrak{X}_1))\cap\mathbb{Q}(\mathfrak{X}_2)$.*
2. *O_1 is faithful if there is an O_1-admissible $x\in\mathbb{Q}(\mathfrak{X}_1)$.*

3. *Suppose a nonnegative semiring $U \subset \mathbb{Q}$ with $1 \in U$. If O_1 is either $\left\{\geq^U_{\mathfrak{X}_1}, >^U_{\mathfrak{X}_1}\right\}$, $\left\{\geq^U_{\mathfrak{X}_1^{\pm 1}}, >^U_{\mathfrak{X}_1^{\pm 1}}\right\}$, or $\left\{\geq_{O_{\mathfrak{X}_1}}, >_{O_{\mathfrak{X}_1}}\right\}$, then O_1 is faithful.*

Proof. Proof of Claim 1. If $f \in A(\succeq_1, \mathbb{Q}(\mathfrak{X}_1)) \cap \mathbb{Q}(\mathfrak{X}_2)$, then $f \in A(\succeq_2, \mathbb{Q}(\mathfrak{X}_2))$ by the reflexivity $f \succeq_1 f$. Claim 1 holds, since the $\succeq_2$-$\succeq_1$ implication yields

$$A(\succeq_2, \mathbb{Q}(\mathfrak{X}_2)) \subset A(\succeq_1, \mathbb{Q}(\mathfrak{X}_1)) \cap \mathbb{Q}(\mathfrak{X}_2).$$

Proof of Claim 2. Claim 2 of Lemma 5.20 gives $1 \succ_1 0$ by $1 \in \mathbb{Z}[x]$. This implies $1 \succ_2 0$ by $\{1,0\} \subset \mathbb{Q}(\mathfrak{X}_2)$. Hence, Claim 2 follows from Claim 1, since $A(\succeq_2, \mathbb{Q}(\mathfrak{X}_2)) = A(\succ_2, \mathbb{Q}(\mathfrak{X}_2))$ by Lemma 5.10.

Proof of Claim 3. Claim 1 gives Claim 3, since each element of $\mathfrak{X}_1$ is O_1-admissible by $1 \in U$. □

By Lemma 7.5, the squaring orders of parcels are always faithful, as parcels have admissible variables. But in general, we have unfaithful squaring orders.

Example 7.6. Let $\mathfrak{X}_1 = \{X_{1,1}, X_{1,2}\}$ and $\mathfrak{X}_2 = \{X_{1,1}\}$. Also, we consider the following binary relations:

- $f \succeq_1 g$ on $\mathbb{Q}(\mathfrak{X}_1)$ if $f, g \in \mathbb{Q}[\mathfrak{X}_2]$, $f - g \in \mathbb{Z}_{\geq 0}[\mathfrak{X}_2]$, and $\mathrm{ord}_{X_{1,1}}(f-g) > 0$;
- $f \succ_1 g$ if $f \succeq_1 g$ and $f \neq g$.

Then, $O_1 = \{\succeq_1, \succ_1\}$ is of squaring orders on $\mathfrak{X}_1$, but not faithful on $\mathfrak{X}_1$.

We verify that O_1 is of squaring orders on $\mathfrak{X}_1$. We first prove that $A(\succeq_1, \mathbb{Q}(\mathfrak{X}_1))$ is a $\succeq_1$-poring. The reflexivity of $\succeq_1$ is by $\mathrm{ord}_{X_{1,1}}(0) = \infty > 0$. The transitivity of $\succeq_1$ holds, since $f \succeq_1 g \succeq_1 h$ implies

$$\mathrm{ord}_{X_{1,1}}(f-h) = \min(\mathrm{ord}_{X_{1,1}}(f-g), \mathrm{ord}_{X_{1,1}}(g-h)) > 0$$

by $f-g, g-h \in \mathbb{Z}_{\geq 0}[\mathfrak{X}_2]$. The antisymmetricity of $\succeq_1$ holds, since $f \succeq_1 g \succeq_1 f$ implies $f = g$ by $f-g, g-f \in \mathbb{Z}_{\geq 0}[\mathfrak{X}_2]$. We also have the additivity and multiplicativity, because we have $f+h \succeq_1 g+h$ for $f \succeq_1 g$ and $h \in A(\succeq_1, \mathbb{Q}(\mathfrak{X}_1))$, and we have $fg \succeq_1 0$ for $f, g \succeq_1 0$ by

$$\mathrm{ord}_{X_{1,1}}(fg) = \mathrm{ord}_{X_{1,1}}(f) + \mathrm{ord}_{X_{1,1}}(g).$$

Hence, $A(\succeq_1, \mathbb{Q}(\mathfrak{X}_1)) = \mathbb{Q}[\mathfrak{X}_2]$ is a $\succeq_1$-poring.

Second, $A(\succeq_1, \mathbb{Q}(\mathfrak{X}_1))$ is also a strict $\succ_1$-poring, since we similarly obtain the transitivity, additivity, and multiplicativity of $\succ_1$.

Third, since we have the squaring implications of O_1 on $\mathfrak{X}_1$, we obtain the semi-strict transitivity of O_1 as follows. Suppose $f \succeq_1 g \succ_1 h$. Then, $f-g, g-h \in \mathbb{Z}_{\geq 0}[\mathfrak{X}_2]$ and $g-h \neq 0$. Hence, $f-h \neq 0$. This gives $f \succ_1 h$, because the $\succ_1$-$\succeq_1$ implication implies $f \succeq_1 g \succeq_1 h$, which gives $f \succeq_1 h$. Similarly, $f \succ_1 g \succeq_1 h$ implies $f \succ_1 h$. Therefore, O_1 consists of squaring orders on $\mathfrak{X}_1$.

However, for $\mathfrak{X}_3 = \{X_{1,2}\}$, let $\succeq_3 = \succeq_1|_{\mathbb{Q}(\mathfrak{X}_3)}$ and $\succ_3 = \succ_1|_{\mathbb{Q}(\mathfrak{X}_3)}$. Then, O_1 is not faithful, since

$$\emptyset = A(\succ_3, \mathbb{Q}(\mathfrak{X}_3)) \neq A(\succeq_3, \mathbb{Q}(\mathfrak{X}_3)) = \mathbb{Q} = \mathbb{Q}[\mathfrak{X}_2] \cap \mathbb{Q}(\mathfrak{X}_3).$$

We have the following squaring orders as restrictions of faithful squaring orders.

Lemma 7.7. *Consider faithful $O_1 = \{\succeq_1, \succ_1\}$ on $\mathfrak{X}_1$. For $\mathfrak{X}_2 = r_{\lambda,\kappa}(\mathfrak{X}_1)$, suppose $\succ_2 = \succ_1|_{\mathbb{Q}(\mathfrak{X}_2)}$ and $\succeq_2 = \succeq_1|_{\mathbb{Q}(\mathfrak{X}_2)}$. Let $O_2 = \{\succeq_2, \succ_2\}$. Then, we have the following.*

1. *For $f,g \in \mathbb{Q}(\mathfrak{X}_2)$, $f >_{\mathrm{O}_{\mathfrak{X}_1}} g$ if and only if $f >_{\mathrm{O}_{\mathfrak{X}_2}} g$. Also, $f \geq_{\mathrm{O}_{\mathfrak{X}_1}} g$ if and only if $f \geq_{\mathrm{O}_{\mathfrak{X}_2}} g$.*

2. *We have squaring orders O_2 on $\mathfrak{X}_2$ such that $\succ_2$ is a strict squaring order of $\succeq_2$.*

3. *If $x \in \mathbb{Q}(\mathfrak{X}_2)$ is O_1-admissible, then x is O_2-admissible.*

Proof. Let $L_i = \#(\mathfrak{X}_i)$. For simplicity, suppose $\lambda = [\![L_2]\!]$ so that $\{X_{1,i}\}_{i\in[\![L_2]\!]} \subset \mathfrak{X}_1 = \{X_{1,i}\}_{i\in[\![L_1]\!]}$ and $\mathfrak{X}_2 = \left\{X_{1,i}^{\kappa_i}\right\}_{i\in[\![L_2]\!]}$.

Proof of Claim 1. We first prove the only if part of the strict inequalities. Fix some $u \in \mathbb{R}$ such that $0 < u < 1$. Then, for $r = (r_i)_{i\in[\![L_2]\!]} \in \mathrm{O}_{\mathfrak{X}_2}$, we put $\mathrm{rad}_{u,\lambda,\kappa}(r) \in \mathrm{O}_{\mathfrak{X}_1}$ such that

$$\mathrm{rad}_{u,\lambda,\kappa}(r)_i = \begin{cases} \sqrt[\kappa_i]{r_i} \text{ if } i \in [\![L_2]\!], \\ v_i = u \text{ if } i \in [\![L_2+1, L_1]\!]. \end{cases}$$

Then, $f >_{\mathrm{O}_{\mathfrak{X}_1}} g$ implies $f(r) > g(r)$ for each $r \in \mathrm{O}_{\mathfrak{X}_2}$ by $f(\mathrm{rad}_{u,\lambda,\kappa}(r)) > g(\mathrm{rad}_{u,\lambda,\kappa}(r))$ for $\mathrm{rad}_{u,\lambda,\kappa}(r) \in \mathrm{O}_{\mathfrak{X}_1}$.

Second, we prove the if part. For $r \in \mathrm{O}_{\mathfrak{X}_1}$, we put

$$\mathrm{pow}_{\lambda,\kappa}(r) = \left(r_i^{\kappa_i}\right)_{i\in[\![L_2]\!]} \in \mathrm{O}_{\mathfrak{X}_2}.$$

Then, $f >_{\mathrm{O}_{\mathfrak{X}_2}} g$ gives $f(r) > g(r)$ for each $r \in \mathrm{O}_{\mathfrak{X}_1}$ by $f(\mathrm{pow}_{\lambda,\kappa}(r)) > g(\mathrm{pow}_{\lambda,\kappa}(r))$ for $\mathrm{pow}_{u,\lambda,\kappa}(r) \in \mathrm{O}_{\mathfrak{X}_2}$. The equivalence for non-strict inequalities holds similarly.

Proof of Claim 2. First, we have the $\succeq_2$-$\succ_2$-poring equality $A(\succeq_2, \mathbb{Q}(\mathfrak{X}_2)) = A(\succ_2, \mathbb{Q}(\mathfrak{X}_2))$, since O_1 is faithful. Also, the semi-strict transitivity holds for O_2, as it holds for O_1.

Second, we prove the $\succeq_2$-$\geq_{\mathrm{O}_{\mathfrak{X}_2}}$-poring inclusion. Let $f \in A(\succeq_2, \mathbb{Q}(\mathfrak{X}_2))$. Suppose $f \succeq_2 g$ by some $g \in \mathbb{Q}(\mathfrak{X}_2)$. Then, $f \succeq_1 g$. We deduce $f \geq_{\mathrm{O}_{\mathfrak{X}_1}} g$ by the half $\succeq_1$-$\geq_{\mathrm{O}_{\mathfrak{X}_1}}$ implication and Claim 1 of Lemma 5.5. It follows that $f \geq_{\mathrm{O}_{\mathfrak{X}_2}} g$ by Claim 1. Similarly, $g \succeq_2 f$ implies $g \geq_{\mathrm{O}_{\mathfrak{X}_2}} f$.

Third, we prove the squaring implications of O_2 on $\mathfrak{X}_2$. By Claim 1, the $>_{\mathrm{O}_{\mathfrak{X}_2}}$-$\geq_{\mathrm{O}_{\mathfrak{X}_2}}$ implication follows from the $>_{\mathrm{O}_{\mathfrak{X}_1}}$-$\geq_{\mathrm{O}_{\mathfrak{X}_1}}$ implication. The $\succ_2$-$\succeq_2$ implication follows from the $\succ_1$-$\succeq_1$ implication. For the half $\succeq_2$-$\geq_{\mathrm{O}_{\mathfrak{X}_2}}$ implication, let $f \succeq_2 0$. Then, since $f \succeq_1 0$, we have $f \geq_{\mathrm{O}_{\mathfrak{X}_2}} 0$ by the half $\succeq_1$-$\geq_{\mathrm{O}_{\mathfrak{X}_1}}$ implication and Claim 1. Similarly, we obtain the half $\succ_2$-$>_{\mathrm{O}_{\mathfrak{X}_2}}$ implication.

Proof of Claim 3. We have the half $>_x$-$\succ_2$ implication, as O_2 restricts O_1 to $\mathbb{Q}(\mathfrak{X}_2)$. We have the upper condition of x on $\mathrm{O}_{\mathfrak{X}_2}$ by $1 >_{\mathrm{O}_{\mathfrak{X}_1}} x$, because $x(r)$ for each $r \in \mathrm{O}_{\mathfrak{X}_2}$ is $x(\mathrm{rad}_{u,\lambda,\kappa}(r)) < 1$. □

Then, the following proposition gives parcels on restricted coordinates.

Proposition 7.8. *Consider $\mathscr{F} = \Lambda(s,l,w,\succ_1,f_s,\phi,\rho,x,\mathfrak{X}_1)$ for squaring orders $O_1 = \{\succeq_1,\succ_1\}$ on $\mathfrak{X}_1$. Let $\mathfrak{X}_2 = r_{\lambda,\kappa}(\mathfrak{X}_1)$. Assume $\zeta = (\mathscr{F},\mathfrak{X}_2)$ with the base and numerator conditions. Let $O_2 = \{\succeq_2,\succ_2\}$ for $\succeq_2 = \succeq_1|_{\mathbb{Q}(\mathfrak{X}_2)}$ and $\succ_2 = \succ_1|_{\mathbb{Q}(\mathfrak{X}_2)}$. Then, we have the following.*

1. *O_2 has squaring orders on $\mathfrak{X}_2$ such that $\succ_2$ is a strict squaring order of $\succeq_2$.*

2. *x is O_2-admissible.*

3. *f_s is $\succ_2$-positive.*

4. *ϕ is a $(s,l,w,\succ_2,\rho,x,\mathfrak{X}_2)$-mediator.*

5. *There is a parcel $\Lambda(s,l,w,\succ_2,f_s,\phi,\rho,x,\mathfrak{X}_2)$.*

Proof. Proof of Claim 1. Since x is O_1-admissible, O_1 is faithful by Claim 2 of Lemma 7.5. This gives Claim 1 by Claim 2 of Lemma 7.7.

Proof of Claim 2. Claim 2 holds by the base condition of ζ and Claim 3 of Lemma 7.7.

Proof of Claim 3. Claim 3 holds by the numerator condition of ζ.

Proof of Claim 4. Let $\mu_i = (s, l, w, \succ_i, \rho, x, \mathfrak{X}_i)$ for $i \in [\![2]\!]$. First, we verify the base positivity of ϕ and μ_2. The base condition of ζ gives $\phi(x) \in \mathbb{Q}(\mathfrak{X}_2)^l$. Then, $\phi(x)_i^{w_i} >_{O_{\mathfrak{X}_2}} 0$ by Claim 1 of Lemma 7.7, since $\phi(x)_i^{w_i} >_{O_{\mathfrak{X}_1}} 0$ by the base positivity of ϕ and μ_1. Second, the base-shift positivity of ϕ and μ_1 implies that of ϕ and μ_2, since $B(s, l, w, m, \phi, \rho, x, \mathfrak{X}_1) \in \mathbb{Q}(\mathfrak{X}_2)$ by the base condition of ζ.

Proof of Claim 5. Claim 5 follows from Claims 1, 2, 3, and 4. □

Remark 7.9. If $\succ_1 = >_x$ and $\succeq_1 = \geq_x$ in Proposition 7.8, then by $x \in \mathbb{Q}(\mathfrak{X}_2)^l$, $f \succ_1 g$ and $f \succeq_1 g$ are equivalent to $f \succ_2 g$ and $f \succeq_2 g$, respectively.

We introduce the following parcels by Proposition 7.8,

Definition 7.10. *Under the assumption of Proposition 7.8, we define the restricted parcel*

$$r_{\lambda,\kappa}(\mathscr{F}, \mathfrak{X}_1) = \Lambda(s, l, w, \succ_2, f_s, \phi, \rho, x, \mathfrak{X}_2).$$

In Definition 7.10, $\succ_2$ on $\mathbb{Q}(\mathfrak{X}_2)$ depends on $\succ_1$ on $\mathbb{Q}(\mathfrak{X}_1)$, as $\succ_2 = \succ_1|_{\mathbb{Q}(\mathfrak{X}_2)}$. However, we have the following for some restricted squaring orders (see Remark 7.9 for $\geq_x, >_x$ of the base x of a parcel).

Proposition 7.11. *Let $\mathfrak{X}_1$ and $\mathfrak{X}_2 = r_{\lambda,\kappa}(\mathfrak{X}_1)$. Then, we have the following:*

$$\geq_{O_{\mathfrak{X}_1}}|_{\mathbb{Q}(\mathfrak{X}_2)} = \geq_{O_{\mathfrak{X}_2}} \text{ and } >_{O_{\mathfrak{X}_1}}|_{\mathbb{Q}(\mathfrak{X}_2)} = >_{O_{\mathfrak{X}_2}}; \tag{7.2.1}$$

$$\geq^U_{\mathfrak{X}_1}|_{\mathbb{Q}(\mathfrak{X}_2)} = \geq^U_{\mathfrak{X}_2} \text{ and } >^U_{\mathfrak{X}_1}|_{\mathbb{Q}(\mathfrak{X}_2)} = >^U_{\mathfrak{X}_2}; \tag{7.2.2}$$

$$\geq^U_{\mathfrak{X}_1^{\pm1}}|_{\mathbb{Q}(\mathfrak{X}_2)} = \geq^U_{\mathfrak{X}_2^{\pm1}} \text{ and } >^U_{\mathfrak{X}_1^{\pm1}}|_{\mathbb{Q}(\mathfrak{X}_2)} = >^U_{\mathfrak{X}_2^{\pm1}}. \tag{7.2.3}$$

Proof. Claim 1 of Lemma 7.7 gives equations (7.2.1). We prove equations (7.2.2). First, let $f \geq^U_{\mathfrak{X}_1} g$ for $f, g \in \mathbb{Q}(\mathfrak{X}_2)$. Then, $f \geq^U_{\mathfrak{X}_2} g$, since $f - g \in U[\mathfrak{X}_1]$ implies $f - g \in U[\mathfrak{X}_2]$ by $\mathbb{Q}[\mathfrak{X}_1] \cap \mathbb{Q}(\mathfrak{X}_2) = \mathbb{Q}[\mathfrak{X}_2]$. Second, if $f \geq^U_{\mathfrak{X}_2} g$ for $f, g \in \mathbb{Q}[\mathfrak{X}_2]$, then $f \geq^U_{\mathfrak{X}_1} g$ by $f - g \in U[\mathfrak{X}_2] \subset U[\mathfrak{X}_1]$. Also, if $f, g \in \mathbb{Q}(\mathfrak{X}_2)$ and $f \neq g$, then $f >^U_{\mathfrak{X}_1} g$ is the same as $f >^U_{\mathfrak{X}_2} g$, since $f >^U_{\mathfrak{X}_1} g$ and $f >^U_{\mathfrak{X}_2} g$ imply $f \geq^U_{\mathfrak{X}_1} g$ and $f \geq^U_{\mathfrak{X}_2} g$, respectively. Therefore, equations (7.2.2) follow. Similarly, equations (7.2.3) hold. □

Since $\mathbb{Q}[\mathfrak{X}]$ is a unique factorization domain, we employ the following notation.

Definition 7.12. *If $f \in \mathbb{Q}(\mathfrak{X})$, then we write some* $\mathrm{Ir}(\mathfrak{X})(f) \in \mathbb{Q}[\mathfrak{X}]^2$ *such that* $\mathrm{Ir}(\mathfrak{X})(f)_1$ *and* $\mathrm{Ir}(\mathfrak{X})(f)_2$ *are coprime and*

$$f = \mathrm{Frac}(\mathrm{Ir}(\mathfrak{X})(f)).$$

We use $\mathrm{Ir}(\mathfrak{X})(f)$ when our argument does not depend on the choice of $\mathrm{Ir}(\mathfrak{X})(f)_1$ and $\mathrm{Ir}(\mathfrak{X})(f)_2$ in $\mathbb{Q}[\mathfrak{X}]$. We state the following lemmas to obtain the optimal parcel coordinate of a parcel.

Lemma 7.13. *Assume $\mathbb{Q}(\mathfrak{X})$ of $\mathfrak{X} = \{X_i\}_{i \in [\![L]\!]}$. Let $\lambda_1, \lambda_2, \lambda_3 \subset [\![L]\!]$ such that $\lambda_3 = \lambda_1 \cap \lambda_2$. For $i \in [\![2]\!]$, suppose $\kappa_i \in \prod_{i \in \lambda_i} \mathbb{Z}_{\geq 1}$ and $\mathfrak{Y}_i = r_{\lambda_i,\kappa_i}(\mathfrak{X})$. Also, let $\kappa_3 = (\mathrm{lcm}(\kappa_{1,i}, \kappa_{2,i}))_{i \in \lambda_3} \in \prod_{i \in \lambda_3} \mathbb{Z}_{\geq 1}$ and $\mathfrak{Y}_3 = r_{\lambda_3,\kappa_3}(\mathfrak{X})$. Then, we have the following.*

1. $\mathbb{Q}(\mathfrak{Y}_1) \cap \mathbb{Q}(\mathfrak{Y}_2) = \mathbb{Q}(\mathfrak{Y}_3)$.

2. *If* $f = \operatorname{Frac}(g) \in \mathbb{Q}(\mathfrak{Y}_3)$ *such that* $g = \operatorname{Ir}(\mathfrak{Y}_1)(f)$ *and* $\deg_{X_p} g_1, \deg_{X_p} g_2 < \kappa_{2,p}$ *for some* $p \in [\![L]\!]$*, then* $f \in \mathbb{Q}(\mathfrak{Y}_1 \setminus \{X_p\})$.

3. *Consider* $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$*. If* $(\mathscr{F},\mathfrak{Y}_1)$ *and* $(\mathscr{F},\mathfrak{Y}_2)$ *satisfy the base and numerator conditions, then so does* $(\mathscr{F},\mathfrak{Y}_3)$.

Proof. By $\mathbb{Q}[\mathfrak{Y}_1] \cap \mathbb{Q}[\mathfrak{Y}_2] \supset \mathfrak{Y}_3$, $\mathbb{Q}(\mathfrak{Y}_1) \cap \mathbb{Q}(\mathfrak{Y}_2) \supset \mathbb{Q}(\mathfrak{Y}_3)$. Conversely, if $f \in \mathbb{Q}(\mathfrak{Y}_1) \cap \mathbb{Q}(\mathfrak{Y}_2)$, then $\det(\operatorname{Ir}(\mathfrak{Y}_1)(f), \operatorname{Ir}(\mathfrak{Y}_2)(f)) = 0$ in the unique factorization domain $\mathbb{Q}[\mathfrak{X}]$. Hence, each irreducible factor of $\operatorname{Ir}(\mathfrak{Y}_1)(f)_i$ is a factor of $\operatorname{Ir}(\mathfrak{Y}_2)(f)_i$ for $i \in [\![2]\!]$. In particular, $f \in \mathbb{Q}(\mathfrak{Y}_3)$, and hence Claim 1 follows. Claim 2 then holds, since $\deg_{X_p} g_1 = \deg_{X_p} g_2 = 0$ by Claim 1. Also, Claim 3 follows from Claim 1. □

If $u_1, u_2 \in \mathbb{Z}_{\geq 1}^L$ for some $L \in \mathbb{Z}_{\geq 1}$, then let $\operatorname{lcm}(u_1,u_2) = (\operatorname{lcm}(u_{1,i},u_{2,i}))_{i \in [\![L]\!]} \in \mathbb{Z}_{\geq 1}^L$.

Lemma 7.14. *Let* $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X}_1)$ *of* $\mathfrak{X}_1 = \{X_{1,i}\}_{i \in [\![L_1]\!]}$*. Then, we have the following.*

1. *There is the smallest non-empty* $\mathfrak{X}_2 \subset \mathfrak{X}_1$ *such that* $(\mathscr{F},\mathfrak{X}_2)$ *has the base and numerator conditions.*

2. *Let* $r_\lambda(\mathfrak{X}_1) = \mathfrak{X}_2$ *in Claim 1 for some* $\lambda \subset [\![L_1]\!]$*. Consider the partial order* $\geq$ *on* $V := \prod_{i \in \lambda} \mathbb{Z}_{\geq 1}$*. Then, there is the largest* $\kappa \in V$ *such that* $(\mathscr{F},\mathfrak{X}_3)$ *of* $\mathfrak{X}_3 = r_{\lambda,\kappa}(\mathfrak{X}_1)$ *has the base and numerator conditions.*

Proof. Proof of Claim 1. Claim 1 follows from Claim 3 of Lemma 7.13 and the finite cardinality of $\mathfrak{X}_1$.

Proof of Claim 2. If there are $u_1,u_2 \in V$ such that $\left(\mathscr{F}, r_{\lambda,u_1}(\mathfrak{X}_1)\right)$ and $\left(\mathscr{F}, r_{\lambda,u_2}(\mathfrak{X}_1)\right)$ have the base and numerator conditions, then $\left(\mathscr{F}, r_{\lambda,\operatorname{lcm}(u_1,u_2)}(\mathfrak{X}_1)\right)$ has the base and numerator conditions by Claim 3 of Lemma 7.13. Hence, suppose the non-existence of the largest κ: i.e., let $\mu_i \in V$ for $i \in \mathbb{Z}_{\geq 1}$ and $\mathfrak{Y}_i = r_{\lambda,\mu_i}(\mathfrak{X}_1)$ with the following three conditions:

- first, $\mu_1 = \iota^\lambda(1)$;
- second, $(\mathscr{F},\mathfrak{Y}_i)$ satisfies the base and numerator conditions for each $i \in \mathbb{Z}_{\geq 1}$;
- third, some $p \in \lambda$ satisfies $\lim_{i \to \infty} \mu_{i,p} = \infty$.

Assume $g \in \mathbb{Q}(\mathfrak{Y}_i)$ for each $i \in \mathbb{Z}_{\geq 1}$. Let $g = \operatorname{Frac}(h)$ for $h = \operatorname{Ir}(\mathfrak{Y}_1)(g)$. The third condition gives $v \in \mathbb{Z}_{\geq 1}$ such that $\deg_{X_{1,p}} h_i < \mu_{v,p}$ for $i \in [\![2]\!]$. This implies $g \in \mathbb{Q}(\mathfrak{Y}_1 \setminus \{X_{1,p}\})$ by Claim 2 of Lemma 7.13. Hence, the base and numerator conditions hold for $\left(\mathscr{F}, \mathfrak{Y}_1 \setminus \{X_{1,p}\}\right)$ against the smallest assumption of $\mathfrak{Y}_1$. □

By the smallest subset and the largest power above, we derive the following optimal parcel coordinates.

Proposition 7.15. *Let* $\mathscr{F} = \Lambda(s,l,w,\succ_1,f_s,\phi,\rho,x,\mathfrak{X}_1)$*. Then, there is* $\mathfrak{X}_2 = r_{\lambda,\kappa}(\mathfrak{X}_1)$ *such that* $\mathfrak{X}_2$ *is optimal for* $r_{\lambda,\kappa}(\mathscr{F},\mathfrak{X}_1) = \Lambda(s,l,w,\succ_2,f_s,\phi,\rho,x,\mathfrak{X}_2)$*. In particular,* $\mathfrak{X}_2$ *is uniquely determined by* $\mathscr{F}$ *and* $\mathfrak{X}_1$.

Proof. Because Lemma 7.14 gives the existence, let us prove the uniqueness. Let $\mathfrak{X}_2' = r_{\lambda',\kappa'}(\mathfrak{X}_1)$ such that $\mathfrak{X}_2'$ is optimal for $r_{\lambda',\kappa'}(\mathscr{F}) = \Lambda(s,l,w,\succ_2',f_s,\phi,\rho,x,\mathfrak{X}_2')$. First, suppose $r_\mu(\mathfrak{X}_1) = r_\lambda(\mathfrak{X}_1) \setminus r_{\lambda'}(\mathfrak{X}_1)$ and $r_{\mu,p}(\mathfrak{X}_1) \subset \mathfrak{X}_2$. By Claim 1 of Lemma 7.13, the base and numerator conditions hold for $\left(\mathscr{F}, r_{\lambda_1 \cap \lambda_1'}(\mathfrak{X}_1)\right)$. Furthermore, each $v \in \mathbb{Z}_{\geq 1}$ gives the base and numerator conditions of $\left(\mathscr{F}, r_{\mu,vp}(\mathfrak{X}_1) \cup \left(\mathfrak{X}_2 \setminus r_{\mu,p}(\mathfrak{X}_1)\right)\right)$. By the optimal property of $\mathfrak{X}_2$, we deduce $\lambda' = \lambda$. Second, Claim 2 of Lemma 7.14 gives $\kappa = \kappa'$. □

We introduce the following notion to explicitly discuss the real values of parcels.

Definition 7.16. *Suppose* $\mathscr{F} = \Lambda(s,l,w,\succ_1,f_s,\phi,\rho,x,\mathfrak{X}_1)$*. We call* $\mathfrak{X}$ *fully optimal for* $\mathscr{F}$ *if* $\mathfrak{X}$ *is optimal and* $x \in \mathbb{Q}(\mathfrak{X})^l$ *is fully admissible.*

7.3 Merged determinants by q-binomial coefficients and base shift functions

We write merged determinants by q-binomial coefficients and base shift functions. For this, we adopt the following notation.

Definition 7.17. *Consider a parcel $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ with $m,n \in \mathbb{Z}^l$, $k \in \mathbb{Z}^{2l}$, $a = \nu(k)$, and $b = \nu(m,n,k)$. Let $y = x^\rho$. Then, in $\mathbb{Q}(\mathfrak{X})$, we define*

$$\Delta_L(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) = f_{s,m} f_{s,n^\vee} B(s,l,w,m,n^\vee,\phi,\rho,x,\mathfrak{X}) \begin{bmatrix} b \\ a \end{bmatrix}_{y^\sqcup}^{w^\sqcup},$$

$$\Delta_R(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) = f_{s,m\boxminus k} f_{s,(n\boxplus k)^\vee} B(s,l,w,m\boxminus k,(n\boxplus k)^\vee,\phi,\rho,x,\mathfrak{X}) \begin{bmatrix} b \\ a^\vee \end{bmatrix}_{y^\sqcup}^{w^\sqcup}.$$

We first write the σ-plus and σ-minus by ν and flips.

Lemma 7.18. *Let $l \in \mathbb{Z}_{\geq 1}$, $m,n \in \mathbb{Z}^l$, and $k \in \mathbb{Z}^{2l}$ with $a = \nu(k)$ and $b = \nu(m,n,k)$. Then, we have*

$$m\boxminus k = (b - a^\vee)[1:l], \tag{7.3.1}$$
$$n\boxplus k = (b - a^\vee)[l+1:2l]. \tag{7.3.2}$$

Proof. We obtain equations (7.3.1) and (7.3.2), since $i \in [\![l]\!]$ gives

$$b_i - a_{2l-i+1} = \sum k[1:i] + m_i - \sum k[1:2l-i+1] = m_i - \sum k[i+1:2l-i+1] = m_i - \sigma(k)_i,$$
$$b_{i+l} - a_{l-i+1} = \sum k[1:i+l] + n_i - \sum k[1:l-i+1] = n_i + \sum k[l-i+2:l+i] = n_i + \sigma(k)_{l-i+1}.$$

□

Then, we have the following general statement on merged determinants. This allows us to examine the merged-log-concavity not only through q-binomial coefficients and base shift functions, but also through general non-negativities and positivities on squaring orders.

Theorem 7.19. *Suppose a parcel $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$. Let $\mu = (s,l,m,n,k)$ for $m,n \in \mathbb{Z}^l$ and $k \in \mathbb{Z}^{2l}$.*

1. *We have the following equations:*

$$\Delta_L(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) = \Upsilon(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) \cdot \mathscr{F}_m \mathscr{F}_{n^\vee}; \tag{7.3.3}$$
$$\Delta_R(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) = \Upsilon(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) \cdot \mathscr{F}_{m\boxminus k} \mathscr{F}_{(n\boxplus k)^\vee}. \tag{7.3.4}$$

2. *We obtain the following equation:*

$$\Delta(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) = \Delta_L(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) - \Delta_R(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}).$$

3. *We have the following inequalities:*

$$\Delta_L(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) \succeq 0; \tag{7.3.5}$$
$$\Delta_R(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) \succeq 0. \tag{7.3.6}$$

4. *Let μ be fitting. Then, we obtain*

$$\Delta_L(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) \succ 0, \tag{7.3.7}$$

and

$$\Delta_R(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) \begin{cases} \succ 0 & \text{if } \mu \text{ is wrapped,} \quad (7.3.8)\\ = 0 & \text{if } \mu \text{ is unwrapped.} \quad (7.3.9)\end{cases}$$

Proof. Let $y = x^\rho$, $a = \nu(k)$, and $b = \nu(m,n,k)$.

Proof of Claim 1. We first prove equation (7.3.3). When $m \mathbin{+\!\!+} n \notin [\![s]\!]^{2l}$ or $a \mathbin{+\!\!+} b \not\geq 0$, it holds by $0 = 0$. In fact, if $m \mathbin{+\!\!+} n \notin [\![s]\!]^{2l}$, then $\mathscr{F}_m = f_{s,m} = 0$ or $\mathscr{F}_{n^\vee} = f_{s,n^\vee} = 0$. Also, if $a \mathbin{+\!\!+} b \not\geq 0$, then $\left[\begin{smallmatrix} b \\ a\end{smallmatrix}\right]^{w^\sqcup}_{y^\sqcup} = \Upsilon(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) = 0$.

Assume $m \mathbin{+\!\!+} n \in [\![s]\!]^{2l}$ and $a \mathbin{+\!\!+} b \geq 0$. Then, $m = (b-a)[1:l] \geq 0$ gives

$$\begin{aligned}
f_{s,m} \cdot B(s,l,w,m,\phi,\rho,x,\mathfrak{X}) \cdot \begin{bmatrix} b[1:l] \\ a[1:l]\end{bmatrix}^w_y &= f_{s,m} \cdot \prod_{i \in [\![l]\!]} \frac{\phi_i(y_i)^{m_i w_i}[m_i]!^{w_i}_{y_i}}{\phi_i(x_i)^{m_i w_i}[m_i]!^{w_i}_{x_i}} \cdot \prod_{i \in [\![l]\!]} \frac{[b_i]!^{w_i}_{y_i}}{[a_i]!^{w_i}_{y_i}\,[m_i]!^{w_i}_{y_i}} \\
&= f_{s,m} \cdot \prod_{i \in [\![l]\!]} \frac{\phi_i(y_i)^{m_i w_i}[b_i]!^{w_i}_{y_i}}{\phi_i(x_i)^{m_i w_i}[a_i]!^{w_i}_{y_i}\,[m_i]!^{w_i}_{x_i}} \\
&= \frac{f_{s,m}}{\prod \phi(x)^{m \circ w} \cdot [m]!^w_x} \cdot \prod \phi(y)^{(b-a)[1:l]\circ w} \cdot \frac{[b[1:l]]!^w_y}{[a[1:l]]!^w_y} \\
&= \mathscr{F}_{s,m} \cdot \prod \phi(y)^{(b-a)[1:l]\circ w} \cdot \frac{[b[1:l]]!^w_y}{[a[1:l]]!^w_y}.
\end{aligned} \tag{7.3.10}$$

Also, $n = (b-a)[l+1:2l] \geq 0$ gives

$$\begin{aligned}
&f_{s,n^\vee} \cdot B(s,l,w,n^\vee,\phi,\rho,x,\mathfrak{X}) \cdot \begin{bmatrix} b[l+1:2l] \\ a[l+1:2l]\end{bmatrix}^{w^\vee}_{y^\vee} \\
&= f_{s,n^\vee} \cdot \prod_{i \in [\![l]\!]} \frac{\phi_{l-i+1}(y_{l-i+1})^{n_i w_{l-i+1}}[n_i]!^{w_{l-i+1}}_{y_{l-i+1}}}{\phi_{l-i+1}(x_{l-i+1})^{n_i w_{l-i+1}}[n_i]!^{w_{l-i+1}}_{x_{l-i+1}}} \cdot \prod_{i \in [\![l]\!]} \frac{[b_{i+l}]!^{w_{l-i+1}}_{y_{l-i+1}}}{[a_{i+l}]!^{w_{l-i+1}}_{y_{l-i+1}}\,[n_i]!^{w_{l-i+1}}_{y_{l-i+1}}} \\
&= f_{s,n^\vee} \cdot \prod_{i \in [\![l]\!]} \frac{\phi_{l-i+1}(y_{l-i+1})^{n_i w_{l-i+1}}[b_{i+l}]!^{w_{l-i+1}}_{y_{l-i+1}}}{\phi_{l-i+1}(x_{l-i+1})^{n_i w_{l-i+1}}[a_{i+l}]!^{w_{l-i+1}}_{y_{l-i+1}}\,[n_i]!^{w_{l-i+1}}_{x_{l-i+1}}} \\
&= \frac{f_{s,n^\vee}}{\prod \phi(x)^{n^\vee \circ w} \cdot [n^\vee]!^w_x} \cdot \prod (\phi(y)^\vee)^{(b-a)[l+1:2l]\circ w^\vee} \cdot \frac{[b[l+1:2l]]!^{w^\vee}_{y^\vee}}{[a[l+1:2l]]!^{w^\vee}_{y^\vee}} \\
&= \mathscr{F}_{s,n^\vee} \cdot \prod (\phi(y)^\vee)^{(b-a)[l+1:2l]\circ w^\vee} \cdot \frac{[b[l+1:2l]]!^{w^\vee}_{y^\vee}}{[a[l+1:2l]]!^{w^\vee}_{y^\vee}}.
\end{aligned} \tag{7.3.11}$$

Equations (7.3.10) and (7.3.11) imply equation (7.3.3) by

$$\frac{[b[1:l]]!^w_y}{[a[1:l]]!^w_y} \cdot \frac{[b[l+1:2l]]!^{w^\vee}_{y^\vee}}{[a[l+1:2l]]!^{w^\vee}_{y^\vee}} = \frac{[b]!^{w^\sqcup}_{y^\sqcup}}{[a]!^{w^\sqcup}_{y^\sqcup}},$$

$$\prod \phi(y)^{(b-a)[1:l]\circ w} \cdot \prod (\phi(y)^\vee)^{(b-a)[l+1:2l]\circ w^\vee} = \prod (\phi(y)^\sqcup)^{(b-a)\circ w^\sqcup}.$$

Second, we prove equation (7.3.4). As before, when $(m\boxminus k)+\!\!+(n\boxplus k)\notin [\![s]\!]^{2l}$ or $a+\!\!+b\not\geq 0$, equation (7.3.4) holds by $0=0$. Instead, assume a wrapped μ with $a+\!\!+b\geq 0$.

By Lemma 7.18, we have $(b-a^{\vee})[1:l]=m\boxminus k$, which is non-negative by $m\boxminus k\in [\![s]\!]^{l}$. Replacing a and m by $a^{\vee}$ and $m\boxminus k$ in equation (7.3.10), we obtain

$$f_{s,m\boxminus k}\cdot B(s,l,w,m\boxminus k,\phi,\rho,x,\mathfrak{X})\cdot \begin{bmatrix} b[1:l] \\ a^{\vee}[1:l]\end{bmatrix}_{y}^{w} = \mathscr{F}_{s,m\boxminus k}\cdot\prod\phi(y)^{(b-a^{\vee})[1:l]\circ w}\cdot\frac{[b[1:l]]!_{y}^{w}}{[a^{\vee}[1:l]]!_{y}^{w}}. \tag{7.3.12}$$

Also, by Lemma 7.18, we have $(b-a^{\vee})[l+1:2l]=n\boxplus k$, which is non-negative by $n\boxplus k\in [\![s]\!]^{l}$. Replacing a and n by $a^{\vee}$ and $n\boxplus k$ in equation (7.3.11), we obtain

$$\begin{aligned} &f_{s,(n\boxplus k)^{\vee}}\cdot B(s,l,w,(n\boxplus k)^{\vee},\phi,\rho,x,\mathfrak{X})\cdot \begin{bmatrix} b[l+1:2l] \\ a^{\vee}[l+1:2l]\end{bmatrix}_{y^{\vee}}^{w^{\vee}} \\ &= \mathscr{F}_{s,(n\boxplus k)^{\vee}}\cdot\prod(\phi(y)^{\vee})^{(b-a^{\vee})[l+1:2l]\circ w^{\vee}}\cdot\frac{[b[l+1:2l]]!_{y^{\vee}}^{w^{\vee}}}{[a^{\vee}[l+1:2l]]!_{y^{\vee}}^{w^{\vee}}}. \end{aligned} \tag{7.3.13}$$

In the right-hand side of equation (7.3.12), we have

$$\frac{[b[1:l]]!_{y}^{w}}{[a^{\vee}[1:l]]!_{y}^{w}}\cdot\frac{[b[l+1:2l]]!_{y^{\vee}}^{w^{\vee}}}{[a^{\vee}[l+1:2l]]!_{y^{\vee}}^{w^{\vee}}}=\frac{[b]!_{y^{\sqcup}}^{w^{\sqcup}}}{[a^{\vee}]!_{y^{\sqcup}}^{w^{\sqcup}}}=\frac{[b]!_{y^{\sqcup}}^{w^{\sqcup}}}{[a]!_{y^{\sqcup}}^{w^{\sqcup}}}, \tag{7.3.14}$$

since $[a^{\vee}]!_{y^{\sqcup}}^{w^{\sqcup}}=\prod_{i\in[\![l]\!]}[a_{2l-i+1}]_{y_i}^{w_i}\cdot\prod_{i\in[\![l]\!]}[a_{l-i+1}]_{y_{l-i+1}}^{w_{l-i+1}}=\prod_{i\in[\![l]\!]}[a_{l+i}]_{y_{l-i+1}}^{w_{l-i+1}}\cdot\prod_{i\in[\![l]\!]}[a_i]_{y_i}^{w_i}=[a]!_{y^{\sqcup}}^{w^{\sqcup}}$. Also, in the right-hand side of equation (7.3.13), we have

$$\prod\phi(y)^{(b-a^{\vee})[1:l]\circ w}\cdot\prod(\phi(y)^{\vee})^{(b-a^{\vee})[l+1:2l]\circ w^{\vee}}=\prod(\phi(y)^{\sqcup})^{(b-a)\circ w^{\sqcup}}. \tag{7.3.15}$$

We deduce equation (7.3.4) from equations (7.3.12), (7.3.13), (7.3.14), and (7.3.15).

Proof of Claim 2. Claim 2 follows from Claim 1.

Proof of Claim 3. We prove inequality (7.3.5) by the following $\succeq$-non-negativities of $f_{s,m}f_{s,n}$, $\begin{bmatrix} b \\ a\end{bmatrix}_{y^{\sqcup}}^{w^{\sqcup}}$, and $B(s,l,w,m,n,\phi,\rho,x,\mathfrak{X})$. We have $f_{s,m}=0$ when $m\notin [\![s]\!]^{l}$. Then, for each $m\in\mathbb{Z}^{l}$, the $\succ$-positivity of f_s and $\succ$-$\succeq$ implication imply

$$f_{s,m}\succeq 0. \tag{7.3.16}$$

Similarly, for each $m,n\in\mathbb{Z}^{l}$, we have

$$B(s,l,w,m,n,\phi,\rho,x,\mathfrak{X})\succeq 0 \tag{7.3.17}$$

by the base-shift positivity of ϕ and $(s,l,w,\succ,\rho,x,\mathfrak{X})$.

Let us prove

$$\begin{bmatrix} b \\ a\end{bmatrix}_{y^{\sqcup}}^{w^{\sqcup}}=\prod_{i\in[\![l]\!]}\begin{bmatrix} b_i \\ a_i\end{bmatrix}_{x_i^{\rho_i}}^{w_i}\cdot\begin{bmatrix} b_{l+i} \\ a_{l+i}\end{bmatrix}_{x_{l-i+1}^{\rho_{l-i+1}}}^{w_{l-i+1}}\succeq 0. \tag{7.3.18}$$

If $b-a \not\geq 0$, $a \not\geq 0$, or $b \not\geq 0$, then $\begin{bmatrix} b \\ a \end{bmatrix}_{y^{\sqcup}}^{w^{\sqcup}} = 0$. Suppose otherwise. Then, $\begin{bmatrix} b_i \\ a_i \end{bmatrix}_{x_i^{\rho_i}}^{w_i} >_{x_i} 0$ and $\begin{bmatrix} b_{l+i} \\ a_{l+i} \end{bmatrix}_{x_{l-i+1}^{\rho_{l-i+1}}}^{w_{l-i+1}} >_{x_{l-i+1}} 0$ for each $i \in [\![l]\!]$. Furthermore, since each x_i is $\succ$-admissible, the multiplicativity of $\succ$ gives

$$\begin{bmatrix} b \\ a \end{bmatrix}_{y^{\sqcup}}^{w^{\sqcup}} \succ 0. \tag{7.3.19}$$

This induces inequality (7.3.18). Therefore, inequality (7.3.5) holds by inequalities (7.3.16), (7.3.17), and (7.3.18). Similarly, inequality (7.3.6) holds.

Proof of Claim 4. We obtain inequality (7.3.7) by the $\succ$-positivity of f_s, the base-shift positivity of ϕ and $(s,l,w,\succ,\rho,x,\mathfrak{X})$, and inequality (7.3.19), since we have the inclusion condition of μ and Claim 2 of Lemma 3.7. Similarly, we obtain inequality (7.3.8), since $b-a^{\vee} \geq 0$ by $(m \boxminus k) \sqcup\!\!\sqcup (n \boxplus k) \in [\![s]\!]^{2l}$ and Lemma 7.18. Also, equation (7.3.9) holds, since $f_{s,m\boxminus k} f_{s,(n\boxplus k)^{\vee}} = 0$ by $(m \boxminus k) \sqcup\!\!\sqcup (n \boxplus k) \notin [\![s]\!]^{2l}$. ☐

7.4 Cut and shift operators

We introduce the notions of cut and shift operators on parcels. They trim and reindex the positive terms of parcels.

7.4.1 Cut operators

Definition 7.20. *Suppose gates $s_1, s_2 \geq 0$ such that $s_{1,1} \leq s_{2,1} \leq s_{2,2} \leq s_{1,2}$. Let $\mathscr{F}_1 = \Lambda(s_1, l, w, \succ, f_{1,s_1}, \phi, \rho, x, \mathfrak{X})$. Then, we define the parcel*

$$\mathrm{C}_{s_1,s_2}(\mathscr{F}_1) = \Lambda(s_2, l, w, \succ, f_{2,s_2}, \phi, \rho, x, \mathfrak{X})$$

such that

$$f_{2,s_2,m} = \begin{cases} f_{1,s_1,m} \text{ if } m \in [\![s_2]\!]^l, \\ 0 \text{ otherwise}. \end{cases}$$

We call C_{s_1,s_2} a cut operator.

Then, we have the following merged-log-concavity on cut operators.

Proposition 7.21. *Let $\mathscr{F}_i = \Lambda(s_i, l, w, \succ, f_{i,s_i}, \phi, \rho, x, \mathfrak{X})$ for $i \in [\![2]\!]$ such that $\mathscr{F}_2 = \mathrm{C}_{s_1,s_2}(\mathscr{F}_1)$. Suppose a fitting $\mu = (s_2, l, m, n, k)$.*

1. *$\Delta(\mathscr{F}_2)(s_2, l, w, m, n, k, \phi, \rho, x, \mathfrak{X}) = \Delta(\mathscr{F}_1)(s_1, l, w, m, n, k, \phi, \rho, x, \mathfrak{X})$ if μ is wrapped.*
2. *$\Delta(\mathscr{F}_2)(s_2, l, w, m, n, k, \phi, \rho, x, \mathfrak{X}) = \Delta_L(\mathscr{F}_1)(s_1, l, w, m, n, k, \phi, \rho, x, \mathfrak{X})$ if μ is unwrapped.*
3. *$\mathscr{F}_2$ is $\succ'$-merged-log-concave if $\mathscr{F}_1$ is $\succ'$-merged-log-concave.*

Proof. Proof of Claim 1. Claim 1 holds by Claim 2 of Theorem 7.19. In fact, we have

$$f_{2,s_2,m} f_{2,s_2,n^{\vee}} B(s_2, l, w, m, n^{\vee}, \phi, \rho, x, \mathfrak{X}) = f_{1,s_1,m} f_{1,s_1,n^{\vee}} B(s_1, l, w, m, n^{\vee}, \phi, \rho, x, \mathfrak{X})$$

by $s_{1,1} \leq s_{2,1} \leq s_{2,2} \leq s_{1,2}$. Also, since $(m \boxminus k) \sqcup\!\!\sqcup (n \boxplus k) \in [\![s_2]\!]^{2l}$, we have

$$\begin{aligned} &f_{2,s_2,m\boxminus k} f_{2,s_2,(n\boxplus k)^{\vee}} B(s_2, l, w, m \boxminus k, (n \boxplus k)^{\vee}, \phi, \rho, x, \mathfrak{X}) \\ &\quad = f_{1,s_1,m\boxminus k} f_{1,s_1,(n\boxplus k)^{\vee}} B(s_1, l, w, m \boxminus k, (n \boxplus k)^{\vee}, \phi, \rho, x, \mathfrak{X}). \end{aligned}$$

Proof of Claim 2. Claim 4 of Theorem 7.19 implies

$$\Delta(\mathscr{F}_2)(s_2,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) = \Delta_L(\mathscr{F}_2)(s_2,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}),$$

which equals to $\Delta_L(\mathscr{F}_2)(s_1,l,w,m,n,k,\phi,\rho,x,\mathfrak{X})$ by $m \mathbin{+\!\!\!+} n \in [\![s_1]\!]^{2l}$.

Proof of Claim 3. We want to prove

$$\Delta(\mathscr{F}_2)(s_2,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) \succ' 0. \tag{7.4.1}$$

Because μ is fitting, (s_1,l,m,n,k) is fitting. Then, $\Delta(\mathscr{F}_1)(s_1,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) \succ' 0$. If μ is wrapped, this inequality gives inequality (7.4.1) by Claim 1. Since $\mathscr{F}_1$ is $\succ'$-merged-log-concave, we have $\succ' \supseteq \succ$. If μ is unwrapped, this compatibility gives inequality (7.4.1) by Claim 2 and Claim 4 of Theorem 7.19. □

7.4.2 Shift operators

Definition 7.22. *Suppose $\mathscr{F}_1 = \Lambda(s_1,l,w,\succ,f_{1,s_1},\phi,\rho,x,\mathfrak{X})$ with $h \in \mathbb{Z}_{\geq 0}$ and $s_2 = s_1 + h$. We define the parcel*

$$\mathrm{S}_h(\mathscr{F}_1) = \Lambda(s_2,l,w,\succ,f_{2,s_2},\phi,\rho,x,\mathfrak{X})$$

such that

$$f_{2,s_2,m} = \begin{cases} f_{1,s_1,m-h} \cdot \dfrac{[m]!^w_x}{[m-h]!^w_x} \text{ for } m \in [\![s_2]\!]^l, \\ 0 \text{ otherwise}. \end{cases}$$

We call S_h a shift operator. Also, let $f_{2,s_2} = \mathrm{S}_h(f_{1,s_1})$.

Notice that $S_h(\mathscr{F}_1)$ is a parcel by $h \in \mathbb{Z}_{\geq 0}$, because for each $m \in [\![s_2]\!]^l$, the $\succ$-admissibility of x and the $\succ$-positivity of f_{1,s_1} imply $f_{2,s_2,m} = f_{1,s_1,m-h} \cdot \frac{[m]!^w_x}{[m-h]!^w_x} \succ 0$, which is the $\succ$-positivity of f_{2,s_2}.

For example, each $m \in [\![s_1]\!]^l + 1$ satisfies

$$\mathrm{S}_1(\mathscr{F}_1)_m = \frac{f_{1,s_1,m-1}[m]^w_x}{\prod \phi(x)^{m \circ w} \cdot [m]!^w_x} = \frac{f_{1,s_1,m-1}}{\prod \phi(x)^{m \circ w} \cdot [m-1]!^w_x}.$$

We first prove $\mathrm{S}_{h+1} = \mathrm{S}_1 \circ \mathrm{S}_h$ to discuss the merged-log-concavity of shift operators.

Lemma 7.23. *Assume $\mathscr{F}_1 = \Lambda(s_1,l,w,\succ,f_{1,s_1},\phi,\rho,x,\mathfrak{X})$ and $h \in \mathbb{Z}_{\geq 0}$. Then, $\mathrm{S}_{h+1}(\mathscr{F}_1) = \mathrm{S}_1(\mathrm{S}_h(\mathscr{F}_1))$.*

Proof. Let $s_2 = s_1 + h$ and $s_3 = s_4 = s_2 + 1$. Consider the parcels

$$\begin{aligned} \mathscr{F}_2 &= \mathrm{S}_h(\mathscr{F}_1) = \Lambda(s_2,l,w,\succ,f_{2,s_2},\phi,\rho,x,\mathfrak{X}), \\ \mathscr{F}_3 &= \mathrm{S}_1(\mathscr{F}_2) = \Lambda(s_3,l,w,\succ,f_{3,s_3},\phi,\rho,x,\mathfrak{X}), \\ \mathscr{F}_4 &= \mathrm{S}_{h+1}(\mathscr{F}_1) = \Lambda(s_4,l,w,\succ,f_{4,s_4},\phi,\rho,x,\mathfrak{X}). \end{aligned}$$

Let $m \in [\![s_4]\!]^l$. The assertion now follows from

$$f_{3,s_3,m} = f_{2,s_2,m-1} \cdot \frac{[m]!^w_x}{[m-1]!^w_x} = f_{1,s_1,m-1-h} \cdot \frac{[m-1]!^w_x}{[m-1-h]!^w_x} \cdot \frac{[m]!^w_x}{[m-1]!^w_x} = f_{1,s_1,m-h-1} \cdot \frac{[m]!^w_x}{[m-h-1]!^w_x} = f_{4,s_4,m}.$$

□

We then state the following compatibility of the fitting condition and shift operators.

Lemma 7.24. *Let $h \in \mathbb{Z}_{\geq 0}$ and $s_2 = s_1 + h$. For $m,n \in \mathbb{Z}^l$ and $k \in \mathbb{Z}^{2l}$, let $\mu_1 = (s_1, l, m-h, n-h, k)$ and $\mu_2 = (s_2, l, m, n, k)$. Then, we have the following.*

1. *μ_2 is fitting if and only if μ_1 is fitting.*

2. *μ_2 is wrapped if and only if μ_1 is wrapped.*

Proof. Let $a_1 = a_2 = \nu(k)$, $b_1 = \nu(m-h, n-h, k)$, and $b_2 = \nu(m,n,k)$.

Proof of Claim 1. Assume that μ_2 is fitting. We have the inclusion condition of μ_1 by $(m-h) \sqcup\!\!\!\sqcup (n-h) \in [\![s_2]\!]^{2l} - h = [\![s_1]\!]^{2l}$. Also, we have the slope conditions of μ_1 by $a_1 = a_2$ and $b_1 = b_2 - h$. It follows that μ_1 is fitting. The converse holds similarly.

Proof of Claim 2. Claim 2 follows, since $(m \boxminus k) \sqcup\!\!\!\sqcup (n \boxplus k) \in [\![s_2]\!]^{2l}$ is equivalent to

$$((m-h) \boxminus k) \sqcup\!\!\!\sqcup ((n-h) \boxplus k) = (m \boxminus k) \sqcup\!\!\!\sqcup (n \boxplus k) - h \in [\![s_2]\!]^{2l} - h.$$

□

We verify the following equations on the change of variables $m \mapsto m-1$ and $n \mapsto n-1$ by q-numbers and q-binomial coefficients.

Lemma 7.25. *Let $l \in \mathbb{Z}_{\geq 1}$, $w \in \mathbb{Z}^l_{\geq 0}$, $\rho \in \mathbb{Z}^l_{\geq 1}$, and $m,n \in \mathbb{Z}^l_{\geq 1}$. Consider $a, b \in \mathbb{Z}^{2l}_{\geq 0}$ such that $b - a = m \sqcup\!\!\!\sqcup n$. For an indeterminate $x \in \mathbb{Q}(\mathfrak{X})^l$, let $y = x^\rho$. Suppose $\phi(x) \in \prod_{i \in [\![l]\!]} \mathbb{Q}(x_i)$ such that $\prod \phi(x) \neq 0$. Then, we have the following equations:*

$$\frac{\prod \phi(y)^{m \circ w} \cdot [m]!^w_y}{\prod \phi(x)^{m \circ w} \cdot [m]!^w_x} \cdot [m]^w_x \cdot \begin{bmatrix} b[1:l] \\ a[1:l] \end{bmatrix}^w_y = [b[1:l]]^w_y \cdot \frac{\prod \phi(y)^{(m-1) \circ w} \cdot [m-1]!^w_y}{\prod \phi(x)^{(m-1) \circ w} \cdot [m-1]!^w_x} \cdot \frac{\prod \phi(y)^w}{\prod \phi(x)^w} \cdot \begin{bmatrix} b[1:l]-1 \\ a[1:l] \end{bmatrix}^w_y;$$

$$\begin{aligned} \frac{\prod \phi(y)^{n^\vee \circ w} \cdot [n^\vee]!^w_y}{\prod \phi(x)^{n^\vee \circ w} \cdot [n^\vee]!^w_x} \cdot [n^\vee]^w_x \cdot \begin{bmatrix} b[l+1:2l] \\ a[l+1:2l] \end{bmatrix}^{w^\vee}_{y^\vee} &= [b[l+1:2l]]^{w^\vee}_{y^\vee} \cdot \frac{\prod \phi(y)^{(n^\vee-1) \circ w} \cdot [n^\vee-1]!^w_y}{\prod \phi(x)^{(n^\vee-1) \circ w} \cdot [n^\vee-1]!^w_x} \\ &\quad \cdot \frac{\prod \phi(y)^w}{\prod \phi(x)^w} \cdot \begin{bmatrix} b[l+1:2l]-1 \\ a[l+1:2l] \end{bmatrix}^{w^\vee}_{y^\vee}. \end{aligned}$$

Proof. Since $(b-a)[1:l] = m \geq 1$, we have the former equation by

$$\begin{aligned} \text{l.h.s.} &= \frac{\prod_{i \in [\![l]\!]} \phi_i(y_i)^{m_i w_i} [m_i]!^{w_i}_{y_i}}{\prod_{i \in [\![l]\!]} \phi_i(x_i)^{m_i w_i} [m_i]!^{w_i}_{x_i}} \cdot \prod_{i \in [\![l]\!]} [m_i]^{w_i}_{x_i} \cdot \frac{[b_i]!^{w_i}_{y_i}}{[a_i]!^{w_i}_{y_i} [m_i]!^{w_i}_{y_i}} \\ &= \frac{\prod \phi(y)^w}{\prod \phi(x)^w} \cdot \frac{\prod_{i \in [\![l]\!]} \phi_i(y_i)^{(m_i-1) w_i} [m_i-1]!^{w_i}_{y_i}}{\prod_{i \in [\![l]\!]} \phi_i(x_i)^{(m_i-1) w_i} [m_i-1]!^{w_i}_{x_i}} \cdot [b_i]^{w_i}_{y_i} \cdot \frac{[b_i-1]!^{w_i}_{y_i}}{[a_i]!^{w_i}_{y_i} [m_i-1]!^{w_i}_{y_i}} \\ &= \text{r.h.s.} \end{aligned}$$

Since $(b-a)[l+1:2l] = n \geq 1$, the latter equation holds similarly by $\frac{[n^\vee]!^w_y}{[n]!^{w^\vee}_{y^\vee}} = \frac{[n^\vee-1]!^w_y}{[n-1]!^{w^\vee}_{y^\vee}}$. □

We derive the following merged-log-concavity on shift operators.

Proposition 7.26. *Consider $\mathscr{F}_i = \Lambda(s_i, l, w, \succ, f_{i,s}, \phi, \rho, x, \mathfrak{X})$ for $i \in [\![2]\!]$ such that $\mathscr{F}_2 = \mathrm{S}_h(\mathscr{F}_1)$. Let $y = x^\rho$. Also, let $\mu_2 = (s_2, l, m, n, k)$ be fitting.*

1. *We have*

$$\Delta(\mathscr{F}_2)(s_2,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) = \prod_{i\in[\![h]\!]} [\nu(m-i+1, n-i+1, k)]_{y^{\sqcup}}^{w^{\sqcup}} \cdot \left(\frac{\prod \phi(y)^w}{\prod \phi(x)^w}\right)^{2h} \cdot \Delta(\mathscr{F}_1)(s_1,l,w,m-h,n-h,k,\phi,\rho,x,\mathfrak{X}).$$

2. *Assume a squaring order $\succ'$ such that*

$$\left(\frac{\prod \phi(y)^w}{\prod \phi(x)^w}\right)^{2} \succ' 0. \tag{7.4.2}$$

Then, $\mathscr{F}_2$ is $\succ'$-merged-log-concave when $\mathscr{F}_1$ is $\succ'$-merged-log-concave.

Proof. Suppose $a = \nu(k)$ and $b = \nu(m, n, k)$.

Proof of Claim 1. By Lemma 7.23, it suffices to prove it for $h = 1$. Then, Claim 2 of Theorem 7.19 gives

$$\begin{aligned}
\Delta(\mathscr{F}_2)(s_2,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) &= f_{2,s_2,m} f_{2,s_2,n^\vee} \cdot B(s_2,l,w,m,n^\vee,\phi,\rho,x,\mathfrak{X}) \begin{bmatrix} b \\ a \end{bmatrix}_{y^{\sqcup}}^{w^{\sqcup}} \\
&\quad - f_{2,s_2,m\boxminus k} f_{2,s_2,(n\boxplus k)^\vee} \cdot B(s_2,l,w,m\boxminus k,(n\boxplus k)^\vee,\phi,\rho,x,\mathfrak{X}) \begin{bmatrix} b \\ a^\vee \end{bmatrix}_{y^{\sqcup}}^{w^{\sqcup}} \\
&= f_{1,s_1,m-1}[m]_x^w f_{1,s_1,(n-1)^\vee}[n^\vee]_x^w \cdot B(s_2,l,w,m,n^\vee,\phi,\rho,x,\mathfrak{X}) \begin{bmatrix} b \\ a \end{bmatrix}_{y^{\sqcup}}^{w^{\sqcup}} \\
&\quad - f_{1,s_1,m\boxminus k-1}[m\boxminus k]_x^w f_{1,s_1,(n\boxplus k-1)^\vee}[(n\boxplus k)^\vee]_x^w \\
&\quad \cdot B(s_2,l,w,m\boxminus k,(n\boxplus k)^\vee,\phi,\rho,x,\mathfrak{X}) \begin{bmatrix} b \\ a^\vee \end{bmatrix}_{y^{\sqcup}}^{w^{\sqcup}}.
\end{aligned} \tag{7.4.3}$$

First, since μ_2 is fitting, we have $b - a = m + n \geq h = 1$ and $a, b \geq 0$ by Claim 2 in Lemma 3.7. Then, in the right-hand side of equation (7.4.3), Lemma 7.25 gives

$$
\begin{aligned}
&[m]^w_x[n^\vee]^w_x B(s_2,l,w,m,n^\vee,\phi,\rho,x,\mathfrak{X})\begin{bmatrix} b \\ a \end{bmatrix}^{w^\sqcup}_{y^\sqcup} \\
&= [m]^w_x[n^\vee]^w_x B(s_2,l,w,m,\phi,\rho,x,\mathfrak{X})B(s_2,l,w,n^\vee,\phi,\rho,x,\mathfrak{X})\begin{bmatrix} b \\ a \end{bmatrix}^{w^\sqcup}_{y^\sqcup} \\
&= \frac{\prod\phi(y)^{m\circ w}\cdot[m]!^w_y}{\prod\phi(x)^{m\circ w}\cdot[m]!^w_x}\cdot[m]^w_x\cdot\begin{bmatrix} b[1:l] \\ a[1:l] \end{bmatrix}^w_y\cdot\frac{\prod\phi(y)^{n^\vee\circ w}\cdot[n^\vee]!^w_y}{\prod\phi(x)^{n^\vee\circ w}\cdot[n^\vee]!^w_x}\cdot[n^\vee]^w_x\cdot\begin{bmatrix} b[l+1:2l] \\ a[l+1:2l] \end{bmatrix}^w_y \\
&= [b[1:l]]^w_y\cdot\frac{\prod\phi(y)^{(m-1)\circ w}\cdot[m-1]!^w_y}{\prod\phi(x)^{(m-1)\circ w}\cdot[m-1]!^w_x}\cdot\frac{\prod\phi(y)^w}{\prod\phi(x)^w}\cdot\begin{bmatrix} b[1:l]-1 \\ a[1:l] \end{bmatrix}^w_y \\
&\cdot[b[l+1:2l]]^{w^\vee}_{y^\vee}\cdot\frac{\prod\phi(y)^{(n^\vee-1)\circ w}\cdot[n^\vee-1]^w_y}{\prod\phi(x)^{(n^\vee-1)\circ w}\cdot[n^\vee-1]^w_x}\cdot\frac{\prod\phi(y)^w}{\prod\phi(x)^w}\cdot\begin{bmatrix} b[l+1:2l]-1 \\ a[l+1:2l] \end{bmatrix}^{w^\vee}_{y^\vee} \\
&= [b]^{w^\sqcup}_{y^\sqcup}\left(\frac{\prod\phi(y)^w}{\prod\phi(x)^w}\right)^2 B(s_1,l,w,m-1,(n-1)^\vee,\phi,\rho,x,\mathfrak{X})\begin{bmatrix} b-1 \\ a \end{bmatrix}^{w^\sqcup}_{y^\sqcup}.
\end{aligned} \tag{7.4.4}
$$

Second, if μ_2 is wrapped, $b-a^\vee=(m\boxminus k)+\!\!\!+\,(n\boxplus k)\geq h=1$ by Lemma 7.18. Then, Lemma 7.25 gives

$$
\begin{aligned}
&[m\boxminus k]^w_x[(n\boxplus k)^\vee]^w_x B(s_2,l,w,m\boxminus k,(n\boxplus k)^\vee,\phi,\rho,x,\mathfrak{X})\begin{bmatrix} b \\ a^\vee \end{bmatrix}^{w^\sqcup}_{x^\sqcup} \\
&= [b]^{w^\sqcup}_{y^\sqcup}\cdot\left(\frac{\prod\phi(y)^w}{\prod\phi(x)^w}\right)^2\cdot B(s_1,l,w,m\boxminus k-1,(n\boxplus k-1)^\vee,\phi,\rho,x,\mathfrak{X})\cdot\begin{bmatrix} b-1 \\ a^\vee \end{bmatrix}^{w^\sqcup}_{y^\sqcup}.
\end{aligned} \tag{7.4.5}
$$

If μ_2 is unwrapped, then so is μ_1 by Claim 2 of Lemma 7.24. By $0=0$, equation (7.4.5) holds.

We deduce Claim 1 from equations (7.4.3), (7.4.4), and (7.4.5), since

$$
\begin{aligned}
&f_{1,s_1,m-1}f_{1,s_1,(n-1)^\vee}B(s_1,l,w,m-1,(n-1)^\vee,\phi,\rho,x,\mathfrak{X})\begin{bmatrix} b-1 \\ a \end{bmatrix}^{w^\sqcup}_{y^\sqcup} \\
&= \Delta_L(\mathscr{F}_1)(s_1,l,w,m-1,n-1,k,\phi,\rho,x,\mathfrak{X}), \\
&f_{1,s_1,m\boxminus k-1}f_{1,s_1,(n\boxplus k-1)^\vee}B(s_1,l,w,m\boxminus k-1,(n\boxplus k-1)^\vee,\phi,\rho,x,\mathfrak{X})\cdot\begin{bmatrix} b-1 \\ a^\vee \end{bmatrix}^{w^\sqcup}_{y^\sqcup} \\
&= \Delta_R(\mathscr{F}_1)(s_1,l,w,m-1,n-1,k,\phi,\rho,x,\mathfrak{X}).
\end{aligned}
$$

Proof of Claim 2. Since $b-a=m+\!\!\!+\,n\geq h$ and $a\geq 0$, $\nu(m-i+1,n-i+1,k)=b-i+1\geq 1$ for $i\in[\![h]\!]$. In particular, $\prod_{i\in[\![h]\!]}[\nu(m-i+1,n-i+1,k)]^{w^\sqcup}_{y^\sqcup}\succ 0$ by Claim 2 of Lemma 5.20. We obtain Claim 2, since Claim 1 and inequality (7.4.2) imply $\Delta(\mathscr{F}_2)(s_2,l,w,m,n,k,\phi,\rho,x,\mathfrak{X})\succ' 0$. □

We state the following corollary for some specific parcel parameters.

Corollary 7.27. *Let $\mathscr{F}_i=\Lambda(s_i,l,w,\succ,f_{i,s},\phi,\rho,x,\mathfrak{X})$ for $i\in[\![2]\!]$ such that $\mathscr{F}_2=\mathrm{S}_h(\mathscr{F}_1)$. Assume one of the following four cases.*

1. *ϕ is the canonical mediator.*

2. $1 \in [\![s_1]\!]$.

3. $w = \iota^l(0)$.

4. $\rho = \iota^l(1)$.

Then, we have

$$\frac{\prod \phi(y)^w}{\prod \phi(x)^w} \succ 0. \tag{7.4.6}$$

In particular, if $\mathscr{F}_1$ is $\succ'$-merged-log-concave, then $\mathscr{F}_2$ is $\succ'$-merged-log-concave.

Proof. First, suppose Case 1. Then, since $\rho \geq 1$, we have

$$\frac{\prod \phi(y)^w}{\prod \phi(x)^w} = \prod_{i \in [\![l]\!]} \frac{(1-x_i^{\rho_i})^{w_i}}{(1-x_i)^{w_i}} = \prod_{i \in [\![l]\!]} [\rho_i]_{x_i}^{w_i} \succ 0$$

by Claim 2 of Lemma 5.20. Second, suppose Case 2. Let $m = \iota^l(1) \in [\![s_1]\!]^l$. Then, since $\phi(x)$ is the mediator of $\mathscr{F}_1$, its base-shift positivity gives

$$0 \prec B(s_1, l, w, m, \phi, \rho, x, \mathfrak{X}) = \prod_{i \in [\![l]\!]} \frac{\phi_i(y_i)^{m_i w_i} [m_i]!_{y_i}^{w_i}}{\phi_i(x_i)^{m_i w_i} [m_i]!_{x_i}^{w_i}} = \frac{\prod \phi(y)^w}{\prod \phi(x)^w}.$$

Third, inequality (7.4.6) holds for the other cases, since $\frac{\prod \phi(y)^w}{\prod \phi(x)^w} = 1 \succ 0$.

The latter statement holds by Claim 2 of Proposition 7.26. □

8 Explicit merged-log-concave parcels

By monomial conditions in Definition 1.10, we construct explicit merged-log-concave parcels for arbitrary gates, widths, base shifts, and positive weights. Also, we discuss several conjectures on merged determinants.

8.1 Base shift functions, shifted x-binomial products, and pre-merged determinants

We state the following lemmas on the base shift functions $b_{\lambda,\rho}(q)$ in Definition 1.7.

Lemma 8.1. *We have*

$$b_{\lambda,\rho}(q) = \begin{cases} \prod_{h \in [\![\lambda]\!]} [\rho]_{q^h} \text{ if } \lambda \in \mathbb{Z}_{\geq 1}, \\ 1 \text{ if } \lambda = 0. \end{cases}$$

Proof. The assertion follows from Lemma 4.2. □

Lemma 8.2. *If $\lambda \in \mathbb{Z}_{\geq 0}$, then $\deg_q b_{\lambda,\rho}(q) = \frac{(\rho-1)\lambda(\lambda+1)}{2}$.*

Proof. By Lemma 8.1, $\deg_q b_{\lambda,\rho}(q) = \deg_q q^{\rho-1} q^{2(\rho-1)} \dots q^{\lambda(\rho-1)} = \frac{(\rho-1)\lambda(\lambda+1)}{2}$. □

Lemma 8.3. *Consider $\lambda \geq k \geq 0$. Then, we have*

$$\frac{b_{\lambda,\rho}(q)}{b_{\lambda-k,\rho}(q)} = \begin{cases} \sum_{j \in [\![0,\rho-1]\!]^k} q^{\sum_{h \in [\![k]\!]} j_h(\lambda-k+h)} \text{ if } k \geq 1, \\ 1 \text{ otherwise}. \end{cases}$$

Proof. By Lemma 8.1, we have $\frac{b_{\lambda,\rho}(q)}{b_{\lambda-k,\rho}(q)} = \prod_{h\in[\![\lambda-k+1,\lambda]\!]}[\rho]_{q^h} = \prod_{h\in[\![k]\!]}[\rho]_{q^{\lambda-k+h}}$ if $k \geq 1$ and $b_{\lambda,\rho}(q) = b_{\lambda-k,\rho}(q) \neq 0$ if $k = 0$. □

We introduce a tuple version of $b_{\lambda,\rho}(q)$, as well as shifted x-binomial products and pre-merged determinants.

Definition 8.4. *Let $\mu \in \mathbb{Z}_{\geq 1}$ and $\lambda = \lfloor \frac{\mu}{2} \rfloor$. Suppose an indeterminate $x \in \mathbb{Q}(\mathfrak{X})^{\mu}$ and $w, \rho \in \mathbb{Z}^{\mu}_{\geq 1}$ such that x, w, ρ are palindromic. Let ϕ be the canonical μ-mediator.*

1. *If $m \in \mathbb{Z}^{\mu}$, then we define the base shift function*

$$B(\mu, w, m, \rho, x) = \prod_{i\in[\![\mu]\!]} b^{w_i}_{m_i,\rho_i}(x_i) \in \mathbb{Q}(\mathfrak{X}).$$

2. *We define the shifted x-binomial product $U(\mu, w, \rho, x) = \left(U^b_a(\mu, w, \rho, x) \in \mathbb{Q}(\mathfrak{X})\right)_{a,b\in\mathbb{Z}^{\mu}}$ such that*

$$U^b_a(\mu, w, \rho, x) = \begin{bmatrix} b \\ a \end{bmatrix}^w_{x^{\rho}} B(\mu, w, b-a, \rho, x).$$

We refer to μ, w, ρ, and x as the width, weight, base shift, and base of $U(\mu, w, \rho, x)$.

3. *If $e \in \mathbb{Z}^{\lambda}_{\geq 0}$, then we define the pre-merged determinant*

$$d(U)^b_a(\mu, w, \rho, e, x) = U^b_a(\mu, w, \rho, x) - U^b_{a^{\vee}}(\mu, w, \rho, x) \cdot \prod x[1:\lambda]^e \in \mathbb{Q}(\mathfrak{X}).$$

We call e the degree shift of $d(U)^b_a(\mu, w, \rho, e, x)$. In particular, if x is flat and $e \in \mathbb{Z}_{\geq 0}$, then we define

$$\tilde{d}(U)^b_a(\mu, w, \rho, e, x) = d(U)^b_a(\mu, w, \rho, (e) \sqcup \iota^{\lambda-1}(0), x).$$

We adopt the following notation to discuss shifted x-binomial products and pre-merged determinants by polynomial degrees.

Definition 8.5. *Let $\mu \in \mathbb{Z}_{\geq 2}$, $\lambda = \lfloor \frac{\mu}{2} \rfloor$, and $a, b, w, \rho \in \mathbb{Z}^{\mu}$. Then, we define*

$$\begin{aligned}
N^b_a(\mu, w, \rho) &= \left(w_i\rho_i(b_{\mu-i+1} - b_i)(a_{\mu-i+1} - a_i)\right)_{i\in[\![\lambda]\!]} \in \mathbb{Z}^{\lambda}, \\
n^b_a(\mu, w, \rho) &= \sum N^b_a(\mu, w, \rho), \\
h^b_a(\mu, w, \rho) &= \left\lfloor \frac{n^b_a(\mu, w, \rho)}{2} \right\rfloor.
\end{aligned}$$

In particular, if $\rho = \iota^{\mu}(1)$, then let $N^b_a(\mu, w) = N^b_a(\mu, w, \rho)$, $n^b_a(\mu, w) = n^b_a(\mu, w, \rho)$, and $h^b_a(\mu, w) = h^b_a(\mu, w, \rho)$.

We have the following degree differences of width-two shifted x-binomial products.

Lemma 8.6. *Let $a, b \in \mathbb{Z}^2$ such that $a, b-a, b-a^{\vee} \geq 0$. Consider flat $w, \rho \in \mathbb{Z}^2_{\geq 1}$ and $x = \iota^2(q) \in \mathbb{Q}(\mathfrak{X})^2$. Then, we have*

$$\deg_q\left(\begin{bmatrix} b \\ a \end{bmatrix}^w_{x^{\rho}}\right) - \deg_q\left(\begin{bmatrix} b \\ a^{\vee} \end{bmatrix}^w_{x^{\rho}}\right) = n^b_a(2, w, \rho), \tag{8.1.1}$$

$$\deg_q(U^b_a(2, w, \rho, x)) - \deg_q(U^b_{a^{\vee}}(2, w, \rho, x)) = n^b_a(2, w). \tag{8.1.2}$$

Proof. By flat w and ρ, we have $\deg_q\left(\begin{bmatrix} b \\ a \end{bmatrix}^{w}_{x^\rho}\right) = \deg_q\left(\begin{bmatrix} b_1 \\ a_1 \end{bmatrix}^{w_1}_{q^{\rho_1}} \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}^{w_1}_{q^{\rho_1}}\right) = \rho_1 w_1(a_1(b_1-a_1)+a_2(b_2-a_2))$ and $\deg_q\left(\begin{bmatrix} b \\ a^\vee \end{bmatrix}^{w}_{x^\rho}\right) = \rho_1 w_1(a_2(b_1-a_2)+a_1(b_2-a_1))$, both of which have $-a_1^2$ and $-a_2^2$.

First, we deduce equation (8.1.1) by $a_1b_1+a_2b_2-a_2b_1-a_1b_2 = (b_2-b_1)(a_2-a_1)$.

Second, equation (8.1.2) holds as follows. By equation (8.1.1), we have

$$\begin{aligned}
\deg_q(U^b_a(2,w,\rho,x)) - \deg_q(U^b_{a^\vee}(2,w,\rho,x)) &= \deg_q\left(\begin{bmatrix} b \\ a \end{bmatrix}^{w}_{x^\rho}\right) + \deg_q(B(2,w,b-a,\rho,x)) \\
&\quad - \deg_q\left(\begin{bmatrix} b \\ a^\vee \end{bmatrix}^{w}_{x^\rho}\right) - \deg_q(B(2,w,b-a^\vee,\rho,x)) \\
&= n^b_a(2,w,\rho) + \deg_q(B(2,w,b-a,\rho,x)) \\
&\quad - \deg_q(B(2,w,b-a^\vee,\rho,x)).
\end{aligned}$$

We obtain equation (8.1.2), since Lemma 8.2 gives

$$\begin{aligned}
&\deg_q(B(2,w,b-a,\rho,x)) - \deg_q(B(2,w,b-a^\vee,\rho,x)) \\
&= \deg_q(b^{w_1}_{b_1-a_1,\rho_1}(q) b^{w_1}_{b_2-a_2,\rho_1}(q)) - \deg_q(b^{w_1}_{b_1-a_2,\rho_1}(q) b^{w_1}_{b_2-a_1,\rho_1}(q)) \\
&= \frac{w_1(\rho_1-1)}{2} \cdot \left(\sum_{i\in[\![2]\!]} (b_i-a_i)(b_i-a_i+1) - (b_i-a_{2-i+1})(b_i-a_{2-i+1}+1)\right) \\
&= -\frac{w_1(\rho_1-1)}{2} \cdot \left(\sum_{i\in[\![2]\!]} (b_i-a_{2-i+1})^2 - (b_i-a_i)^2\right) \\
&= -(\rho_1-1)w_1(b_2-b_1)(a_2-a_1).
\end{aligned}$$

□

We adopt the following notation to discuss the shifted x-binomial products and pre-merged determinants of general widths.

Definition 8.7. *Let $\mu \in \mathbb{Z}_{\geq 3}$ and $a \in \mathbb{Z}^\mu$. Also, suppose $\lambda = \lfloor \frac{\mu}{2} \rfloor$ and $e \in \mathbb{Z}^\lambda$. Let $\chi(\mu) = 2$ if μ is even and 1 otherwise. Then, we define the following tuples:*

$$\begin{aligned}
\mathrm{O}(a) &= a[1:\lambda-\chi(\mu)+1] +\!\!+ a[\lambda+2:\mu] \in \mathbb{Z}^{\mu-\chi(\mu)}; \\
\mathrm{C}(a) &= a[\lambda-\chi(\mu)+2:\lambda+1] \in \mathbb{Z}^{\chi(\mu)}; \\
\mathrm{O}(\mu,e) &= e[1:\lambda-\chi(\mu)+1] \in \mathbb{Z}^{\lambda-\chi(\mu)+1}; \\
\mathrm{C}(\mu,e) &= e[\lambda:\lambda] \in \mathbb{Z}^{\chi(\mu)-1} \text{ if } \chi(\mu)=2.
\end{aligned}$$

We call $\mathrm{O}(a)$ and $\mathrm{C}(a)$ outer and center tuples of a. Also, we call $\mathrm{O}(\mu,e)$ and $\mathrm{C}(\mu,e)$ outer and center tuples of e.

Then, we have the following lemma for shifted x-binomial products and pre-merged determinants.

Lemma 8.8. *For $\mu \in \mathbb{Z}_{\geq 3}$, suppose $a, b \in \mathbb{Z}^\mu$. Let $\lambda = \lfloor \frac{\mu}{2} \rfloor$.*

1. We have the following equations:

$$U^b_a(\mu,w,\rho,x) = U^{\mathrm{O}(b)}_{\mathrm{O}(a)}(\mu-\chi(\mu),\mathrm{O}(w),\mathrm{O}(\rho),\mathrm{O}(x))\cdot U^{\mathrm{C}(b)}_{\mathrm{C}(a)}(\chi(\mu),\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x));$$
$$U^b_{a^\vee}(\mu,w,\rho,x) = U^{\mathrm{O}(b)}_{\mathrm{O}(a)^\vee}(\mu-\chi(\mu),\mathrm{O}(w),\mathrm{O}(\rho),\mathrm{O}(x))\cdot U^{\mathrm{C}(b)}_{\mathrm{C}(a)^\vee}(\chi(\mu),\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x)).$$

2. In particular, if μ is odd, then

$$d(U)^b_a(\mu,\rho,w,e,x) = U^{\mathrm{C}(b)}_{\mathrm{C}(a)}(\chi(\mu),\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x))\cdot d(U)^{\mathrm{O}(b)}_{\mathrm{O}(a)}(\mu-\chi(\mu),\mathrm{O}(v),\mathrm{O}(\rho),\mathrm{O}(\mu,e),\mathrm{O}(x));$$

also, if μ is even, then

$$\begin{aligned} d(U)^b_a(\mu,\rho,w,e,x) = {} & U^{\mathrm{C}(b)}_{\mathrm{C}(a)}(\chi(\mu),\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x))\cdot U^{\mathrm{O}(b)}_{\mathrm{O}(a)}(\mu-\chi(\mu),\mathrm{O}(w),\mathrm{O}(\rho),\mathrm{O}(x)) \\ & -\prod x[\lambda:\lambda]^{\mathrm{C}(\mu,e)}\cdot U^{\mathrm{C}(b)}_{\mathrm{C}(a)^\vee}(\chi(\mu),\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x)) \\ & \cdot\prod x[1:\lambda-1]^{\mathrm{O}(\mu,e)}\cdot U^{\mathrm{O}(b)}_{\mathrm{O}(a)^\vee}(\mu-\chi(\mu),\mathrm{O}(w),\mathrm{O}(\rho),\mathrm{O}(x)). \end{aligned}$$

Proof. Proof of Claim 1. Since $\mathrm{O}(w)$, $\mathrm{O}(\rho)$, $\mathrm{O}(x)$, and $\mathrm{C}(w)$, $\mathrm{C}(\rho)$, $\mathrm{C}(x)$ are palindromic, each factor in equations of Claim 1 exists. Also,

$$a = \mathrm{O}(a)[1:\lambda-\chi(\mu)+1] \mathbin{+\!\!+} \mathrm{C}(a) \mathbin{+\!\!+} \mathrm{O}(a)[\lambda-\chi(\mu)+2:\mu-\chi(\mu)],$$
$$b = \mathrm{O}(b)[1:\lambda-\chi(\mu)+1] \mathbin{+\!\!+} \mathrm{C}(b) \mathbin{+\!\!+} \mathrm{O}(b)[\lambda-\chi(\mu)+2:\mu-\chi(\mu)].$$

We obtain Claim 1 by $\mathrm{O}(a^\vee) = \mathrm{O}(a)^\vee$ and $\mathrm{C}(a^\vee) = \mathrm{C}(a)^\vee$.

Proof of Claim 2. Claim 2 holds by Claim 1, since $C(a) = C(a)^\vee$ for odd μ. □

We introduce the following notion to later discuss merged determinants by pre-merged ones.

Definition 8.9. *Let $\mu \in \mathbb{Z}_{\geq 2}$ and $a,b \in \mathbb{Z}^\mu$. Consider a tuple $\omega = (\mu,a,b)$.*

1. We call ω pre-fitting if ω satisfies the following conditions:

(a) a is non-negative and increasing;

(b) b is increasing;

(c) $b_1 < b_\mu$ and $a_1 < a_\mu$;

(d) $a \leq b$.

We refer to Condition 1c as the end slope condition of ω.

2. We call ω tempered if $a^\vee \leq b$.

3. If $\mu \geq 3$, then we define the outer tuple $\tilde{\mathrm{O}}(\omega) = (\mu-\chi(\mu),\mathrm{O}(a),\mathrm{O}(b))$.

For example, if (μ,a,b) is pre-fitting and tempered, then $0 \leq a_1 \leq a_2 \leq \ldots \leq a_\mu \leq b_1 \leq b_2 \leq \ldots \leq b_\mu$. We adopt the term "pre-fitting" by the following statement.

Proposition 8.10. *Suppose a gate $s \geq 0$, $l \in \mathbb{Z}_{\geq 1}$, $m,n \in \mathbb{Z}^l$, and $k \in \mathbb{Z}^{2l}$. Let $\omega_1 = (s,l,m,n,k)$, $a = \nu(k)$, $b = \nu(m,n,k)$, and $\omega_2 = (2l,a,b)$. Then, we have the following.*

1. *If ω_1 is fitting, then ω_2 is pre-fitting.*

2. *If ω_1 is wrapped, then ω_2 is tempered.*

Proof. Proof of Claim 1. Claim 1 follows from Claim 2 of Lemma 3.7, because the slope conditions of ω_1 give the end slope condition of a and b.

Proof of Claim 2. Claim 2 holds by $s \geq 0$, since $b - a^\vee = (m \boxminus k) + (n \boxplus k) \in [\![s]\!]^{2l}$ by Lemma 7.18. □

The converse of Claim 1 in Proposition 8.10 does not hold, because $b - a \in [\![s]\!]^{2l}$ is not necessarily true. Likewise, the converse of Claim 2 in Proposition 8.10 does not hold.

Furthermore, we state the following for outer tuples $\tilde{\mathrm{O}}(\omega)$.

Lemma 8.11. *Let $\mu \in \mathbb{Z}_{\geq 3}$, $a, b \in \mathbb{Z}^\mu$, and $\omega = (\mu, a, b)$.*

1. *If ω is pre-fitting, then $\tilde{\mathrm{O}}(\omega)$ is pre-fitting.*

2. *If ω is tempered, then $\tilde{\mathrm{O}}(\omega)$ is tempered.*

Proof. Proof of Claim 1. Claim 1 holds, since a and $\mathrm{O}(a)$ (or b and $\mathrm{O}(b)$) have the same end terms.

Proof of Claim 2. Claim 2 follows from $\mathrm{O}(b) \geq \mathrm{O}(a^\vee) = \mathrm{O}(a)^\vee$. □

We observe that equation (8.1.2) in Lemma 8.6 is independent on base shifts ρ. This independence extends to the following shifted x-binomial products of general widths.

Proposition 8.12. *Suppose a tempered pre-fitting $\omega = (\mu, a, b)$. Let $x = \iota^\mu(q)$. Then,*

$$\deg_q(U_a^b(\mu, w, \rho, x)) - \deg_q(U_{a^\vee}^b(\mu, w, \rho, x)) = n_a^b(\mu, w), \tag{8.1.3}$$

$$n_a^b(\mu, w) > 0. \tag{8.1.4}$$

Proof. We prove equation (8.1.3) by induction on μ. Suppose $\mu = 2$. Then, w and ρ are flat. Lemma 8.6 gives equation (8.1.3), since a, $b - a$, and $b - a^\vee$ are non-negative for the tempered pre-fitting ω.

Suppose $\mu > 2$. Since ω is tempered, $U_a^b(\mu, w, \rho, x)$ and $U_{a^\vee}^b(\mu, w, \rho, x)$ are non-zero for non-negative a, $b - a$, and $b - a^\vee$. Also, $\tilde{\mathrm{O}}(\omega) = (\mu - \chi(\mu), \mathrm{O}(a), \mathrm{O}(b))$ is tempered pre-fitting by Lemma 8.11. In particular, $\mathrm{C}(a) = \mathrm{C}(a)^\vee$ for odd μ. Then, the induction on μ and Claim 1 of Lemma 8.8 give equation (8.1.3) by

$$\begin{aligned}\deg_q(U_a^b(\mu, w, \rho, x)) - \deg_q(U_{a^\vee}^b(\mu, w, \rho, x)) &= \begin{cases} n_{\mathrm{O}(a)}^{\mathrm{O}(b)}(\mu - \chi(\mu), \mathrm{O}(w)) \text{ if } \mu \text{ is odd,} \\ n_{\mathrm{O}(a)}^{\mathrm{O}(b)}(\mu - \chi(\mu), \mathrm{O}(w)) + n_{\mathrm{C}(a)}^{\mathrm{C}(b)}(\chi(\mu), \mathrm{C}(w)) \text{ otherwise,} \end{cases} \\ &= n_a^b(\mu, w).\end{aligned}$$

Since a and b are increasing, the end slope condition of ω implies inequality (8.1.4) by

$$n_a^b(\mu, w) = \sum_{i \in [\![\lfloor \frac{\mu}{2} \rfloor]\!]} w_i (b_{\mu - i + 1} - b_i)(a_{\mu - i + 1} - a_i) \geq w_1 (b_\mu - b_1)(a_\mu - a_1) > 0.$$

□

8.2 Positivity of pre-merged determinants

We obtain polynomials with positive integer coefficients by pre-merged determinants. For this, we recall the following notions of q-polynomials.

Definition 8.13. *Suppose $f(q) = \sum_{i \in \mathbb{Z}_{\geq 0}} f_i q^i \in \mathbb{Q}[q]$ and $c(f) = (f_i)_{i \in [\![\mathrm{ord}_q(f)]\!], \deg_q(f))}$. The q-polynomial f is called palindromic if $c(f)$ is palindromic, unimodal if $c(f)$ is unimodal, and log-concave if $c(f)$ is log-concave. Also, f is said to have a step if $c(f)$ has a step.*

Hence, for example, $f(q) \in \mathbb{Q}[q]$ is unimodal if and only if $f = 0$ or $f_{\mathrm{ord}_q(f)} \leq f_{\mathrm{ord}_q(f)+1} \leq \dots \leq f_i \geq \dots \geq f_{\deg_q(f)-1} \geq f_{\deg_q(f)}$ for some $i \in \mathbb{Z}$. We now recall the well-known statement below on palindromic and unimodal q-polynomials.

Proposition 8.14. *([Sta, Proposition 1]) For each palindromic and unimodal q-polynomials $f(q), g(q) \in \mathbb{Q}_{\geq 0}[q]$, the following is a palindromic and unimodal q-polynomial:*

$$f(q)g(q) \in \mathbb{Q}_{\geq 0}[q].$$

The following extends Proposition 8.14 for the change of variable $q \mapsto q^\rho$ of $\rho \in \mathbb{Z}_{\geq 1}$. This is to construct explicit merged-log-concave parcels for non-trivial base shifts.

Proposition 8.15. *Suppose $\rho \in \mathbb{Z}_{\geq 1}$ and $\lambda(q) = [\rho]_q$. Then, we have the following.*

1. *For each palindromic and unimodal q-polynomial $h(q) \in \mathbb{Q}_{\geq 0}[q]$ and palindromic and unimodal q^ρ-polynomial $f(q) \in \mathbb{Q}_{\geq 0}[q^\rho]$, the following is a palindromic and unimodal q-polynomial:*

$$f(q)\lambda(q)h(q).$$

2. *$\lambda(q)$ is the unique lowest-degree q-polynomial such that $\lambda(0) = 1$ with the above property.*

Proof. Proof of Claim 1. First, a product of palindromic q-polynomials is a palindromic q-polynomial, since palindromic q-polynomials $\phi(q), \psi(q) \in \mathbb{Q}[q]$ such that $\phi, \psi \neq 0$ imply

$$\sum_{j_1+j_2=i} \phi_{j_1}\psi_{j_2} = \sum_{j_1+j_2=i} \phi_{\deg_q(\phi)-j_1+\mathrm{ord}_q(\phi)}\psi_{\deg_q(\psi)-j_2+\mathrm{ord}_q(\psi)} = \sum_{j_1+j_2=\deg_q(\phi)+\deg_q(\psi)-i+\mathrm{ord}_q(\phi)+\mathrm{ord}_q(\psi)} \phi_{j_1}\psi_{j_2}.$$

Second, we prove that $\mu(q) = f(q)\lambda(q)$ is a unimodal q-polynomial for $\rho > 1$, since $\mu(q)$ is a palindromic q-polynomial by the above. This gives Claim 1 by Proposition 8.14. Let $\lambda(q) = \sum_{0 \leq i \leq \rho-1} \lambda_i q^i$ and $\mu(q) = f(q)\lambda(q) = \sum_{0 \leq i \leq \deg_q(f)+\rho-1} \mu_i q^i$. Assume $\mathrm{ord}_q(f) = 0$ for simplicity. Then, we need

$$\mu_{i+1} - \mu_i \geq 0 \tag{8.2.1}$$

when $0 \leq i \leq \frac{\deg_q(f)+\rho-1}{2} - 1$. Since $\mu_{i+1} - \mu_i = \sum_j f_{i-j}(\lambda_{j+1} - \lambda_j)$, we have

$$\mu_{i+1} - \mu_i = f_{i+1} - f_{i+1-\rho}. \tag{8.2.2}$$

Suppose

$$\frac{\deg_q(f)}{2} - 1 < i \leq \frac{\deg_q(f)+\rho-1}{2} - 1.$$

Since $\deg_q(f) = \rho k$ for some $k \in \mathbb{Z}$, we have

$$\rho k < 2i+2 \leq \rho(k+1) - 1.$$

This gives $i+1 \not\equiv 0 \pmod{\rho}$. Since $f_{i+1} = f_{i+1-\rho} = 0$ and equation (8.2.2), we deduce inequality (8.2.1) by $\mu_{i+1} - \mu_i = 0$. Suppose

$$0 \leq i \leq \frac{\deg_q(f)}{2} - 1.$$

We then deduce inequality (8.2.1) by equation (8.2.2), since $f(q)$ is a palindromic and unimodal q-polynomial.

Proof of Claim 2. Let $\lambda'(q) \in \mathbb{Q}[q]$ such that $\lambda'(0) = 1$ and $\deg_q(\lambda'(q)) \leq \rho - 1$. Suppose that $(1 + q^{\rho})\lambda'(q)$ is a palindromic and unimodal q-polynomial. Since $\lambda'(q)$ has no steps and $\deg_q \lambda'(q) = \rho - 1$, Claim 2 follows. □

We introduce the following rational functions to analyze pre-merged determinants.

Definition 8.16. *Let $a, b \in \mathbb{Z}^2$ and $w, \rho \in \mathbb{Z}^2_{\geq 1}$. Then, we define*

$$\chi_{a,b}(w,\rho,q) = \begin{bmatrix} b \\ a \end{bmatrix}^{w}_{q^{\rho_1}} [\rho_1]_q \in \mathbb{Z}_{\geq 0}[q],$$

$$\kappa_{a,b}(w,\rho,q) = \frac{\prod_{i \in [\![2]\!]} b^{w_i}_{b_i - a_2, \rho_1}(q)}{[\rho_1]_q} \in \mathbb{Q}(q).$$

Each q-binomial coefficient is a palindromic and unimodal q-polynomial [Oha, Syl]. This gives the following for $\chi_{a,b}(w,\rho,q)$ by Claim 1 of Proposition 8.15.

Corollary 8.17. *If $b \geq a \geq 0$, then $\chi_{a,b}(w,\rho,q) >_q 0$ is a palindromic and unimodal q-polynomial.*

Proof. Since $b \geq a \geq 0$, $\begin{bmatrix} b \\ a \end{bmatrix}_{q^{\rho_1}} >_q 0$. This leads to $\chi_{a,b}(w,\rho,q) >_q 0$ by $[\rho_1]_q >_q 0$. Also, Claim 1 of Proposition 8.15 implies that $\chi_{a,b}(w,\rho,q)$ is a palindromic and unimodal q-polynomial, since $\begin{bmatrix} b_1 \\ a_1 \end{bmatrix}_{q^{\rho_1}}$ and $\begin{bmatrix} b_2 \\ a_2 \end{bmatrix}_{q^{\rho_1}}$ are palindromic and unimodal q^{ρ_1}-polynomials. □

Also, we have the following positivity of $\kappa_{a,b}(w,\rho,q)$.

Lemma 8.18. *If $a, b \in \mathbb{Z}^2$ satisfy $b_2 > a_2$ and $b_1 \geq a_2$, then $\kappa_{a,b}(w,\rho,q) >_q 0$.*

Proof. The assertion holds, since Lemma 8.1 gives $\frac{b^{w_2}_{b_2 - a_2, \rho_1}(q)}{[\rho_1]_q} = \frac{\prod_{h \in [\![b_2 - a_2]\!]} [\rho_1]^{w_2}_{q^h}}{[\rho_1]_q} >_q 0$ and $b^{w_1}_{b_1 - a_2, \rho_1}(q) >_q 0$. □

Then, the following positivity holds by width-two pre-merged determinants.

Proposition 8.19. *For a tempered pre-fitting $\omega = (2, a, b)$, consider a flat $w \in \mathbb{Z}^2_{\geq 1}$ and $e \in [\![0, n^b_a(2,w)]\!]$. Let $x = \iota^2(q)$. Then, we have*

$$\tilde{d}(U)^b_a(2, w, \rho, e, x) >_q 0. \tag{8.2.3}$$

Proof. Since ω is tempered pre-fitting, we have

$$0 \leq a_1 < a_2 \leq b_1 < b_2. \tag{8.2.4}$$

First, for $\rho_1 = 1$ and $e = 0$, we prove inequality (8.2.3) by inequality (8.2.4). We assume $w_1 = 1$, because inequality (8.2.5) implies $\begin{bmatrix} b \\ a \end{bmatrix}_q >_q \begin{bmatrix} b \\ a^\vee \end{bmatrix}_q$, and hence multiplying the both sides give inequality (8.2.3) for $w_1 \geq 1$. By the induction b_2, we want

$$\tilde{d}(U)^b_a(2, w, \rho, e, x) = \begin{bmatrix} b \\ a \end{bmatrix}_q - \begin{bmatrix} b \\ a^\vee \end{bmatrix}_q >_q 0. \tag{8.2.5}$$

The smallest possible b_2 is 2 by inequality (8.2.4). Then, $a_1 = 0$, $a_2 = 1$, and $b_1 = 1$. This yields inequality (8.2.5) by the direct computation:

$$\tilde{d}(U)^b_a(2, w, \rho, e, x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q - \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q \begin{bmatrix} 2 \\ 0 \end{bmatrix}_q = q >_q 0. \tag{8.2.6}$$

Let $b_2 \geq 2$. By the q-Pascal identity, the first and second terms of inequality (8.2.5) give

$$\begin{aligned}
\begin{bmatrix} b \\ a \end{bmatrix}_q &= \left(\begin{bmatrix} b_1 - 1 \\ a_1 - 1 \end{bmatrix}_q + q^{a_1} \begin{bmatrix} b_1 - 1 \\ a_1 \end{bmatrix}_q \right) \cdot \left(\begin{bmatrix} b_2 - 1 \\ a_2 - 1 \end{bmatrix}_q + q^{a_2} \begin{bmatrix} b_2 - 1 \\ a_2 \end{bmatrix}_q \right) \\
&= \begin{bmatrix} b_1 - 1 \\ a_1 - 1 \end{bmatrix}_q \begin{bmatrix} b_2 - 1 \\ a_2 - 1 \end{bmatrix}_q + q^{a_1} \begin{bmatrix} b_1 - 1 \\ a_1 \end{bmatrix}_q \begin{bmatrix} b_2 - 1 \\ a_2 - 1 \end{bmatrix}_q \\
&\quad + q^{a_2} \begin{bmatrix} b_1 - 1 \\ a_1 - 1 \end{bmatrix}_q \begin{bmatrix} b_2 - 1 \\ a_2 \end{bmatrix}_q + q^{a_1 + a_2} \begin{bmatrix} b_1 - 1 \\ a_1 \end{bmatrix}_q \begin{bmatrix} b_2 - 1 \\ a_2 \end{bmatrix}_q; \\
\begin{bmatrix} b \\ a^\vee \end{bmatrix}_q &= \begin{bmatrix} b_1 - 1 \\ a_2 - 1 \end{bmatrix}_q \begin{bmatrix} b_2 - 1 \\ a_1 - 1 \end{bmatrix}_q + q^{a_2} \begin{bmatrix} b_1 - 1 \\ a_2 \end{bmatrix}_q \begin{bmatrix} b_2 - 1 \\ a_1 - 1 \end{bmatrix}_q \\
&\quad + q^{a_1} \begin{bmatrix} b_1 - 1 \\ a_2 - 1 \end{bmatrix}_q \begin{bmatrix} b_2 - 1 \\ a_1 \end{bmatrix}_q + q^{a_1 + a_2} \begin{bmatrix} b_1 - 1 \\ a_2 \end{bmatrix}_q \begin{bmatrix} b_2 - 1 \\ a_1 \end{bmatrix}_q.
\end{aligned}$$

Comparing the powers of q, we prove the following inequalities:

$$\begin{bmatrix} b - 1 \\ a - 1 \end{bmatrix}_q - \begin{bmatrix} b - 1 \\ a^\vee - 1 \end{bmatrix}_q \geq_q 0; \tag{8.2.7}$$

$$\begin{bmatrix} b - 1 \\ (a_1, a_2 - 1) \end{bmatrix}_q - \begin{bmatrix} b - 1 \\ (a_1, a_2 - 1)^\vee \end{bmatrix}_q \geq_q 0; \tag{8.2.8}$$

$$\begin{bmatrix} b - 1 \\ (a_1 - 1, a_2) \end{bmatrix}_q - \begin{bmatrix} b - 1 \\ (a_1 - 1, a_2)^\vee \end{bmatrix}_q \geq_q 0; \tag{8.2.9}$$

$$\begin{bmatrix} b - 1 \\ a \end{bmatrix}_q - \begin{bmatrix} b - 1 \\ a^\vee \end{bmatrix}_q >_q 0. \tag{8.2.10}$$

In particular, inequality (8.2.10) gives the strictness of inequality (8.2.5).

We deduce inequality (8.2.7) by the induction for $a_1 \geq 1$ and by $0 \geq_q 0$ for $a_1 = 0$. Also, inequality (8.2.8) follows by the induction for $a_2 - 1 > a_1$ and by $0 \geq_q 0$ for $a_2 - 1 = a_1$.

Assume $b_1 - 1 < a_2$. Then, $\begin{bmatrix} b_1-1 \\ a_2 \end{bmatrix}_q = 0$. This gives inequality (8.2.9) by $\begin{bmatrix} b-1 \\ (a_1-1,a_2) \end{bmatrix}_q \geq_q 0$. Also, inequality (8.2.10) holds, because $\begin{bmatrix} b-1 \\ a \end{bmatrix}_q >_q 0$ by $b-1 \geq a$ in inequality (8.2.4).

Instead, assume $b_1 - 1 \geq a_2$. Then, inequality (8.2.9) holds by the induction for $a_1 \geq 1$ and by $0 \geq_q 0$ for $a_1 = 0$. Also, inequality (8.2.10) follows from the induction. Therefore, we obtain inequality (8.2.5) for $\rho_1 = 1$ and $e = 0$.

Second, we prove inequality (8.2.3) for $\rho_1 \geq 1$ and $e \in [\![0, n_a^b(2,w)]\!]$. By inequality (8.2.4), we have $d = b - a_1 \in \mathbb{Z}^2$ and $k = a_2 - a_1 \in \mathbb{Z}$ such that

$$d > 0, \tag{8.2.11}$$

$$k > 0. \tag{8.2.12}$$

For $\lambda \in \mathbb{Z}^k$ and $\alpha, \beta \in \mathbb{Z}$, let

$$E(\lambda, w, \alpha, \beta, k) = \beta + w_1 \sum_{i \in [\![k]\!]} \lambda_i(\alpha - k + i) \in \mathbb{Z}.$$

Then, by $d - k = b - a_2$ and Lemma 8.3, we have

$$\begin{aligned}
&\tilde{d}(U)_a^b(2,w,\rho,e,x) \\
&= b^{w_1}_{b_1-a_2,\rho_1}(q) b^{w_1}_{b_2-a_2,\rho_1}(q) \left(\begin{bmatrix} b \\ a \end{bmatrix}^w_{q^{\rho_1}} \frac{b^{w_1}_{b_1-a_1,\rho_1}(q)}{b^{w_1}_{b_1-a_2,\rho_1}(q)} - q^e \begin{bmatrix} b \\ a^\vee \end{bmatrix}^w_{q^{\rho_1}} \frac{b^{w_1}_{b_2-a_1,\rho_1}(q)}{b^{w_1}_{b_2-a_2,\rho}(q)} \right) \\
&= \kappa_{a,b}(w,\rho,q) \left(\chi_{a,b}(w,\rho,q) \frac{b^{w_1}_{b_1-a_1,\rho_1}(q)}{b^{w_1}_{b_1-a_2,\rho_1}(q)} - q^e \chi_{a^\vee,b}(w,\rho,q) \frac{b^{w_1}_{b_2-a_1,\rho_1}(q)}{b^{w_1}_{b_2-a_2,\rho_1}(q)} \right) \\
&= \kappa_{a,b}(w,\rho,q) \left(\chi_{a,b}(w,\rho,q) \sum_{\lambda \in [\![0,\rho_1-1]\!]^k} q^{E(\lambda,w,d_1,0,k)} - \chi_{a^\vee,b}(w,\rho,q) \sum_{\lambda \in [\![0,\rho_1-1]\!]^k} q^{E(\lambda,w,d_2,e,k)} \right).
\end{aligned} \tag{8.2.13}$$

For $\lambda \in [\![0, \rho_1 - 1]\!]^k$, we prove the following inequality:

$$\chi_{a,b}(w,\rho,q) q^{E(\lambda,w,d_1,0,k)} - \chi_{a^\vee,b}(w,\rho,q) q^{E(\lambda,w,d_2,e,k)} >_q 0. \tag{8.2.14}$$

By Lemma 8.18 and equation (8.2.13), inequality (8.2.3) follows from inequality (8.2.14). By the change of variable $q \mapsto q^{\rho_1}$, inequality (8.2.5) implies

$$\begin{bmatrix} b \\ a \end{bmatrix}^w_{q^{\rho_1}} >_{q^{\rho_1}} \begin{bmatrix} b \\ a^\vee \end{bmatrix}^w_{q^{\rho_1}}.$$

Multiplying $[\rho_1]_q$ on the both sides, we induce

$$\chi_{a,b}(w,\rho,q) - \chi_{a^\vee,b}(w,\rho,q) >_q 0. \tag{8.2.15}$$

Furthermore, Lemma 8.6 gives

$$\begin{aligned}
\deg_q(\chi_{a,b}(w,\rho,q)) - \deg_q(\chi_{a^\vee,b}(w,\rho,q)) &= \deg_q \left(\begin{bmatrix} b \\ a \end{bmatrix}^w_{y^\rho} [\rho]_q \right) - \deg_q \left(\begin{bmatrix} b \\ a^\vee \end{bmatrix}^w_{y^\rho} [\rho]_q \right) \\
&= \deg_q \begin{bmatrix} b \\ a \end{bmatrix}^w_{y^\rho} - \deg_q \begin{bmatrix} b \\ a^\vee \end{bmatrix}^w_{y^\rho} \\
&= n_a^b(2,w,\rho).
\end{aligned} \tag{8.2.16}$$

Also,

$$\operatorname{ord}_q(\chi_{a,b}(w,\rho,q)) - \operatorname{ord}_q(\chi_{a^\vee,b}(w,\rho,q)) = 0. \tag{8.2.17}$$

Then, we prove the following inequality:

$$0 \leq E(\lambda,w,d_2,e,k) - E(\lambda,w,d_1,0,k) \leq n_a^b(2,w,\rho). \tag{8.2.18}$$

By Corollary 8.17 with equation (8.2.15) and inequalities (8.2.16) and (8.2.17), inequality (8.2.14) follows from inequality (8.2.18). The left-hand side of inequality (8.2.18) holds by $d_2 - d_1 = b_2 - b_1 > 0$ and $\lambda \geq 0$. The right-hand side of inequality (8.2.18) holds as follows. Since $0 \leq e \leq n_a^b(2,w) = w_1(b_2-b_1)(a_2-a_1)$, inequalities (8.2.11) and (8.2.12) give

$$\begin{aligned} E(\lambda,w,d_2,e,k) - E(\lambda,w,d_1,0,k) &= e + w_1(d_2-d_1)\sum \lambda_i \\ &\leq e + w_1(d_2-d_1)k(\rho_1-1) \\ &\leq w_1\rho_1(b_2-b_1)(a_2-a_1) \\ &= n_a^b(2,w,\rho). \end{aligned}$$

□

Remark 8.20. Suppose $w_1 = \rho_1 = 1$ and $e = 0$. For non-negative integers $a_1 < a_2 \leq b_1 < b_2$, Proposition 8.19 restricts to the following q-polynomials with positive integer coefficients:

$$\tilde{d}(U)_a^b(2,w,\rho,e,x) = \begin{bmatrix} b_1 \\ a_1 \end{bmatrix}_q \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}_q - \begin{bmatrix} b_1 \\ a_2 \end{bmatrix}_q \begin{bmatrix} b_2 \\ a_1 \end{bmatrix}_q >_q 0,$$

which are important for us to obtain almost strictly unimodal sequences.

The strict inequality above differs from the q-log-concavity of q-binomial coefficients in the pioneering works [But, Kra, Sag]. Let us explain more precisely. Suppose non-negative integers $b_1 \leq b_2$, $a_1 \leq a_2$, k, and $\lambda \leq d(a,b,k) := k(2((a_2-a_1)+k)+(b_2-b_1))$. Then, [Kra, Corollary 3] gives

$$T(a,b,k,\lambda,q) := \begin{bmatrix} b_1 \\ a_2 \end{bmatrix}_q \begin{bmatrix} b_2 \\ a_1 \end{bmatrix}_q - q^\lambda \begin{bmatrix} b_1 \\ a_2+k \end{bmatrix}_q \begin{bmatrix} b_2 \\ a_1-k \end{bmatrix}_q \geq_q 0.$$

However, unlike $\tilde{d}(U)_a^b(2,w,\rho,e,x) = q$ in inequality (8.2.6), $\deg_q T(a,b,k,\lambda,q) > 0$ implies that $T(a,b,k,\lambda,q)$ is not a monomial. Indeed, if $\left[\begin{smallmatrix} b_1 \\ a_2 \end{smallmatrix}\right]_q \left[\begin{smallmatrix} b_2 \\ a_1 \end{smallmatrix}\right]_q \left[\begin{smallmatrix} b_1 \\ a_2+k \end{smallmatrix}\right]_q \left[\begin{smallmatrix} b_2 \\ a_1-k \end{smallmatrix}\right]_q \neq 0$, then

$$\deg_q\left(\begin{bmatrix} b_1 \\ a_2 \end{bmatrix}_q \begin{bmatrix} b_2 \\ a_1 \end{bmatrix}_q\right) - \deg_q\left(\begin{bmatrix} b_1 \\ a_2+k \end{bmatrix}_q \begin{bmatrix} b_2 \\ a_1-k \end{bmatrix}_q\right) = d(a,b,k) > 1.$$

Hence, $T(a,b,k,\lambda,q)$ is not a monomial, since $\left[\begin{smallmatrix} b_1 \\ a_2 \end{smallmatrix}\right]_q \left[\begin{smallmatrix} b_2 \\ a_1 \end{smallmatrix}\right]_q$ and $\left[\begin{smallmatrix} b_1 \\ a_2+k \end{smallmatrix}\right]_q \left[\begin{smallmatrix} b_2 \\ a_1-k \end{smallmatrix}\right]_q$ are palindromic and unimodal q-polynomials such that $\operatorname{ord}_q\left(\left[\begin{smallmatrix} b_1 \\ a_2 \end{smallmatrix}\right]_q \left[\begin{smallmatrix} b_2 \\ a_1 \end{smallmatrix}\right]_q\right) = \operatorname{ord}_q\left(\left[\begin{smallmatrix} b_1 \\ a_2+k \end{smallmatrix}\right]_q \left[\begin{smallmatrix} b_2 \\ a_1-k \end{smallmatrix}\right]_q\right) = 0$. If $\left[\begin{smallmatrix} b_1 \\ a_2 \end{smallmatrix}\right]_q \left[\begin{smallmatrix} b_2 \\ a_1 \end{smallmatrix}\right]_q = 0$, then $\left[\begin{smallmatrix} b_1 \\ a_2+k \end{smallmatrix}\right]_q \left[\begin{smallmatrix} b_2 \\ a_1-k \end{smallmatrix}\right]_q = 0$. Also, if $\left[\begin{smallmatrix} b_1 \\ a_2+k \end{smallmatrix}\right]_q \left[\begin{smallmatrix} b_2 \\ a_1-k \end{smallmatrix}\right]_q = 0$, then $\deg_q T(a,b,k,\lambda,q) > 0$ implies that $T(a,b,k,\lambda,q)$ is not a monomial, as it is a palindromic and unimodal q-polynomial.

We extend Proposition 8.19 after the following lemma.

Lemma 8.21. *Consider a shifted x-binomial product $U_a^b(\mu,w,\rho,x)$ with $\lambda=\lfloor\frac{\mu}{2}\rfloor$ and $e\in\mathbb{Z}_{\geq 0}^{\lambda}$. Let $x\in\mathbb{Q}(\mathfrak{X})^{\mu}$ be $\succ$-admissible. Then, we have the following.*

1. *$U_a^b(\mu,w,\rho,x)>_x 0$ if and only if $b\geq a\geq 0$.*
2. *$U_a^b(\mu,w,\rho,x)\geq_x 0$ and $U_a^b(\mu,w,\rho,x)\cdot\prod x[1:\lambda]^e\geq_x 0$.*

Proof. Proof of Claim 1. Unless $b\geq a\geq 0$, $\begin{bmatrix}b\\a\end{bmatrix}^w_{x^\rho}=0$. We deduce Claim 1, since $\begin{bmatrix}b\\a\end{bmatrix}^w_{x^\rho}>_x 0$ by $b\geq a\geq 0$ and $B(\mu,w,b-a,\rho,x)>_x 0$ by Theorem 5.23 and Lemma 8.1.

Proof of Claim 2. Claim 2 holds by $\prod x[1:\lambda]^e\geq_x 0$. □

Theorem 8.22. *Suppose a pre-fitting $\omega=(\mu,a,b)$. Consider a $\succ$-admissible $x\in\mathbb{Q}(\mathfrak{X})^{\mu}$ and $w,\rho\in\mathbb{Z}_{\geq 1}^{\mu}$ such that x, w, and ρ are palindromic. For $\lambda=\lfloor\frac{\mu}{2}\rfloor$, let $e\in\mathbb{Z}_{\geq 0}^{\lambda}$ such that $e\leq N_a^b(\mu,w)$. Then, we have*

$$d(U)_a^b(\mu,w,\rho,e,x)>_x 0. \tag{8.2.19}$$

Proof. Assume that ω is not tempered. Then, the assertion follows from Claim 1 of Lemma 8.21, because $U_{a^\vee}^b(\mu,w,\rho,x)=0$ by $b\not\geq a^\vee$ and $d(U)_a^b(\mu,w,\rho,e,x)=U_a^b(\mu,w,\rho,x)>_x 0$ by $b\geq a\geq 0$.

Hence, assume that ω is tempered. We prove inequality (8.2.19) by the induction on μ. When $\mu=2$, Proposition 8.19 gives inequality (8.2.19). Suppose $\mu\geq 3$. Then, $\mathrm{O}(\mu,e)\leq N_{\mathrm{O}(a)}^{\mathrm{O}(b)}(\mu-\chi(\mu),\mathrm{O}(w))$. Also, $\tilde{\mathrm{O}}(\omega)$ is tempered and pre-fitting by Lemma 8.11. Hence, the induction implies

$$d(U)_{\mathrm{O}(a)}^{\mathrm{O}(b)}(\mu-\chi(\mu),\mathrm{O}(w),\mathrm{O}(\rho),\mathrm{O}(\mu,e),\mathrm{O}(x))>_x 0. \tag{8.2.20}$$

Now, since $b\geq a\geq 0$ implies $\mathrm{C}(b)\geq\mathrm{C}(a)\geq 0$, Claim 1 of Lemma 8.21 gives

$$U_{\mathrm{C}(a)}^{\mathrm{C}(b)}(\chi(\mu),\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x))>_x 0. \tag{8.2.21}$$

If μ is odd, then inequalities (8.2.20) and (8.2.21) give inequality (8.2.19) by Claim 2 of Lemma 8.8. Instead, assume that μ is even. First, let $\mathrm{C}(a)$ or $\mathrm{C}(b)$ be flat. Then, $0\leq\mathrm{C}(\mu,e)\leq N_{\mathrm{C}(a)}^{\mathrm{C}(b)}(\chi(\mu),\mathrm{C}(w))=\rho_\lambda w_\lambda(b_{\lambda+1}-b_\lambda)(a_{\lambda+1}-a_\lambda)=0$. Also, since $\mathrm{C}(x)$, $\mathrm{C}(w)$, and $\mathrm{C}(\rho)$ are flat, inequality (8.2.21) gives

$$\begin{aligned}\prod x[\lambda:\lambda]^{\mathrm{C}(\mu,e)}\cdot U_{\mathrm{C}(a)^\vee}^{\mathrm{C}(b)}(\chi(\mu),\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x))&=\begin{bmatrix}\mathrm{C}(b)\\\mathrm{C}(a)^\vee\end{bmatrix}^{\mathrm{C}(w)}_{\mathrm{C}(x)^{\mathrm{C}(\rho)}}b^{w_\lambda}_{b_\lambda-a_{\lambda+1},\rho_\lambda}(x_\lambda)b^{w_{\lambda+1}}_{b_{\lambda+1}-a_\lambda,\rho_{\lambda+1}}(x_{\lambda+1})\\&=\begin{bmatrix}\mathrm{C}(b)\\\mathrm{C}(a)\end{bmatrix}^{\mathrm{C}(w)}_{\mathrm{C}(x)^{\mathrm{C}(\rho)}}b^{w_\lambda}_{b_\lambda-a_\lambda,\rho_\lambda}(x_\lambda)b^{w_{\lambda+1}}_{b_{\lambda+1}-a_{\lambda+1},\rho_{\lambda+1}}(x_{\lambda+1})\\&=U_{\mathrm{C}(a)}^{\mathrm{C}(b)}(\chi(\mu),\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x))>_x 0.\end{aligned}$$

Hence, inequality (8.2.19) follows from inequality (8.2.20) and Claim 2 of Lemma 8.8. Second, let $0\leq\mathrm{C}(a)_1<\mathrm{C}(a)_2\leq\mathrm{C}(b)_1<\mathrm{C}(b)_2$. Then, $d(U)_{\mathrm{C}(a)}^{\mathrm{C}(b)}(\chi(\mu),\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(\mu,e),\mathrm{C}(x))>_x 0$ by the induction. Hence, we have

$$U_{\mathrm{C}(a)}^{\mathrm{C}(b)}(\chi(\mu),\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x))>_x\prod x[\lambda:\lambda]^{\mathrm{C}(\mu,e)}\cdot U_{\mathrm{C}(a)^\vee}^{\mathrm{C}(b)}(\chi(\mu),\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x)).$$

Also, inequality (8.2.21) implies

$$U_{\mathrm{O}(a)}^{\mathrm{O}(b)}(\mu-\chi(\mu),\mathrm{O}(w),\mathrm{O}(\rho),\mathrm{O}(x))>_x\prod x[1:\lambda-1]^{\mathrm{O}(\mu,e)}\cdot U_{\mathrm{O}(a)^\vee}^{\mathrm{O}(b)}(\mu-\chi(\mu),\mathrm{O}(w),\mathrm{O}(\rho),\mathrm{O}(x)).$$

Now, inequality (8.2.19) follows from Claim 2 of Proposition 5.6, Claim 2 of Lemma 8.8, and Claim 2 of Lemma 8.21. □

In particular, if $x = \iota^{\mu}(q)$, then Theorem 8.22 gives $d(U)_a^b(\mu, w, \rho, e, x) >_q 0$.

8.3 Merged-log-concavity by functional monomial indices

We give merged-log-concave parcels by the following σ-difference functions and functional monomial indices.

Definition 8.23. *Assume a gate $s \geq 0$, $l \in \mathbb{Z}_{\geq 1}$, and $w \in \mathbb{Z}^l_{\geq 1}$. Suppose a function $t : \mathbb{Z}^l \to \mathbb{Q}$.*

1. *For $m, n \in \mathbb{Z}^l$ and $k \in \mathbb{Z}^{2l}$, we define the σ-difference function*

$$t_\Delta(m,n,k) = t(m \boxminus k) + t((n \boxplus k)^\vee) - t(m) - t(n^\vee) \in \mathbb{Q}.$$

2. *We call $\upsilon = (s, l, w, t)$ a functional monomial index if υ satisfies*

$$t_\Delta(m,n,k) \in \mathbb{Z}, \tag{8.3.1}$$

$$0 \leq t_\Delta(m,n,k) \leq n_a^b(2l, w) \tag{8.3.2}$$

for each wrapped fitting (s, l, m, n, k) with $a = \nu(k)$ and $b = \nu(m, n, k)$. We call s, l, w, and t the gate, width, weight, and core function of υ. We refer to (8.3.1) *and* (8.3.2) *as the integer monomial condition and the sum monomial condition of υ.*

We also define the following shifted x-binomial products, quasi-merged determinants, and proper mediators.

Definition 8.24. *Suppose a gate $s \geq 0$, $l \in \mathbb{Z}_{\geq 1}$, and $w, \rho \in \mathbb{Z}^l_{\geq 1}$. Let $\mu = (s, l, w, \succ, \rho, x, \mathfrak{X})$ for a $\succ$-admissible $x \in \mathbb{Q}(\mathfrak{X})^l$. Consider a μ-mediator ϕ.*

1. *We define the shifted x-binomial product $V(s,l,w,\phi,\rho,x) = \left(V_a^b(s,l,w,\phi,\rho,x) \in \mathbb{Q}(\mathfrak{X})\right)_{a,b \in \mathbb{Z}^{2l}}$ such that*

$$V_a^b(s,l,w,\phi,\rho,x) = \begin{bmatrix} b \\ a \end{bmatrix}^{w^\sqcup}_{(x^\rho)^\sqcup} B(s, 2l, w^\sqcup, b - a, \phi^\sqcup, \rho^\sqcup, x^\sqcup, \mathfrak{X}).$$

We refer to s, l, w, ϕ, ρ, and x as the gate, width, weight, mediator, base shift, and base of $V(s,l,w,\phi,\rho,x)$.

2. *Let $a, b \in \mathbb{Z}^{2l}$ and $e \in \mathbb{Z}^l_{\geq 0}$. We define the quasi-merged determinant*

$$d(V)_a^b(s,l,w,\phi,\rho,e,x) = V_a^b(s,l,w,\phi,\rho,x) - V_{a^\vee}^{b}(s,l,w,\phi,\rho,x) \cdot \prod x^e \in \mathbb{Q}(\mathfrak{X}).$$

We call e the degree shift of $d(V)_a^b(s,l,w,\phi,\rho,e,x)$.

3. *We call ϕ a proper μ-mediator (or a proper mediator for short) if*

$$d(V)_a^b(s,l,w,\phi,\rho,e,x) \succ 0$$

for each fitting (s,l,m,n,k), $a = \nu(k)$, $b = \nu(m,n,k)$, and $e \in \mathbb{Z}^l_{\geq 0}$ such that $e \leq N_a^b(2l, w^\sqcup)$.

4. *If x is flat and $e \in \mathbb{Z}_{\geq 0}$, then let*

$$\tilde{d}(V)_a^b(s,l,w,\phi,\rho,e,x) = d(V)_a^b(s,l,w,\phi,\rho,(e) + \iota^{l-1}(0), x).$$

Let us compare pre-merged and quasi-merged determinants.

Lemma 8.25. *Let $s=(0,\infty)$, $l\in\mathbb{Z}_{\geq 1}$, and $e\in\mathbb{Z}^l_{\geq 0}$. Consider a $\succ$-admissible $x\in\mathbb{Q}(\mathfrak{X})^l$ and the canonical l-mediator ϕ. Then, $d(V)^b_a(s,l,w,\phi,\rho,e,x)=d(U)^b_a(2l,w^{\sqcup},\rho^{\sqcup},e,x^{\sqcup})$.*

Proof. Since $\phi(x)$ is canonical,

$$B(2l,w^{\sqcup},b-a,\rho^{\sqcup},x^{\sqcup})=B(s,l,w,(b-a)[1:l],((b-a)[l+1:2l])^{\vee},\phi,\rho,x,\mathfrak{X}),$$
$$B(2l,w^{\sqcup},b-a^{\vee},\rho^{\sqcup},x^{\sqcup})=B(s,l,w,(b-a^{\vee})[1:l],((b-a^{\vee})[l+1:2l])^{\vee},\phi,\rho,x,\mathfrak{X}).$$

These equations give the assertion, since

$$U^b_a(2l,w^{\sqcup},\rho^{\sqcup},x^{\sqcup})=\begin{bmatrix}b\\a\end{bmatrix}^{w^{\sqcup}}_{(x^{\rho})^{\sqcup}}B(2l,w^{\sqcup},b-a,\rho^{\sqcup},x^{\sqcup})=V^b_a(s,l,w,\phi,\rho,x),$$
$$U^b_{a^{\vee}}(2l,w^{\sqcup},\rho^{\sqcup},x^{\sqcup})=\begin{bmatrix}b\\a^{\vee}\end{bmatrix}^{w^{\sqcup}}_{(x^{\rho})^{\sqcup}}B(2l,w^{\sqcup},b-a^{\vee},\rho^{\sqcup},x^{\sqcup})=V^b_{a^{\vee}}(s,l,w,\phi,\rho,x).$$

□

We demonstrate the existence of proper mediators.

Proposition 8.26. *Suppose $s_1=(0,\infty)$, $l\in\mathbb{Z}_{\geq 1}$, and $w,\rho\in\mathbb{Z}^l_{\geq 1}$. Let $\mu_1=(s_1,l,w,\succ,\rho,x,\mathfrak{X})$ for a $\succ$-admissible $x\in\mathbb{Q}(\mathfrak{X})^l$. Let $\mu_2=(s_2,l,w,\succ,\rho,x,\mathfrak{X})$ for a gate $s_2\geq 0$. Consider a μ_1-mediator ϕ. Then, we have the following.*

1. *If ϕ is the canonical l-mediator, then ϕ is a proper μ_1-mediator.*
2. *If $\rho=\iota^l(1)$, then ϕ is a proper μ_1-mediator.*
3. *If ϕ is a proper μ_1-mediator, then ϕ is a proper μ_2-mediator.*

Proof. Proof of Claim 1. Consider a fitting (s_1,l,m,n,k), $a=\nu(k)$, and $b=\nu(m,n,k)$. Then, $(2l,a,b)$ is pre-fitting by Claim 1 of Proposition 8.10. For each $e\in\mathbb{Z}^l_{\geq 0}$ such that $e\leq N^b_a(2l,w^{\sqcup})$, Theorem 8.22 and Lemma 8.25 give the properness of ϕ:

$$d(V)^b_a(s_1,l,w,\phi,\rho,e,x)=d(U)^b_a(2l,w^{\sqcup},\rho^{\sqcup},e,x^{\sqcup})\succ 0.$$

Proof of Claim 2. Claim 2 holds by Theorem 8.22, since Claim 1 of Lemma 4.2 implies

$$B(s_1,2l,w^{\sqcup},b-a,\phi^{\sqcup},\rho^{\sqcup},x^{\sqcup},\mathfrak{X})=B(s_1,2l,w^{\sqcup},b-a^{\vee},\phi^{\sqcup},\rho^{\sqcup},x^{\sqcup},\mathfrak{X})=1.$$

Proof of Claim 3. Suppose a fitting (s_2,l,m,n,k) with $a=\nu(k)$ and $b=\nu(m,n,k)$. Then, we have

$$d(V)^b_a(s_2,l,w,\phi,\rho,e,x)=\begin{cases}d(V)^b_a(s_1,l,w,\phi,\rho,e,x)\succ 0 \text{ if } b-a^{\vee}\in[\![s_2]\!]^{2l},\\ \begin{bmatrix}b\\a\end{bmatrix}^{w^{\sqcup}}_{(x^{\rho})^{\sqcup}}B(s_2,2l,w^{\sqcup},b-a,\phi^{\sqcup},\rho^{\sqcup},x^{\sqcup},\mathfrak{X})\succ 0 \text{ otherwise,}\end{cases}$$

where the latter holds by the base-shift positivity of ϕ and μ_1. □

Let us write merged-determinants by quasi-merged determinants.

Lemma 8.27. *Suppose a parcel $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\rho,\phi,x,\mathfrak{X})$. Assume a fitting (s,l,m,n,k) with $a = \nu(k)$ and $b = \nu(m,n,k)$. Let $y = x^{\rho}$. Then, we have*

$$\begin{aligned}
&(f_{s,m}f_{s,n^{\vee}})^{-1}\Delta(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X})\\
&= B(s,l,w,(b-a)[1:l],((b-a)[l+1:2l])^{\vee},\phi,\rho,x,\mathfrak{X})\begin{bmatrix} b \\ a \end{bmatrix}_{y^{\sqcup}}^{w^{\sqcup}}\\
&-(f_{s,m}f_{s,n^{\vee}})^{-1}f_{s,m\boxminus k}f_{s,(n\boxplus k)^{\vee}}B(s,l,w,(b-a^{\vee})[1:l],((b-a^{\vee})[l+1:2l])^{\vee},\phi,\rho,x,\mathfrak{X})\begin{bmatrix} b \\ a^{\vee} \end{bmatrix}_{y^{\sqcup}}^{w^{\sqcup}}.
\end{aligned}$$

Proof. Claim 2 of Theorem 7.19 and Lemma 7.18 yield the assertion, since $f_{s,m}$ and $f_{s,n^{\vee}}$ are invertible by the $\succ$-positivity. □

We now obtain the following merged-log-concave parcels by functional monomial indices.

Theorem 8.28. *Let $\upsilon = (s,l,w,t)$ be a functional monomial index. Let $x = \iota^{l}(q)$ for a $\succ$-admissible $q \in \mathbb{Q}(\mathfrak{X})$. For $\mu = (s,l,w,>_{q},\rho,x,\mathfrak{X})$, suppose a proper μ-mediator ϕ. Consider a parcel $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ such that*

$$f_{s,m} = \begin{cases} q^{t(m)} \text{ for } m \in [\![s]\!]^{l}, \\ 0 \text{ otherwise}. \end{cases}$$

Then, for each fitting (s,l,m,n,k) with $a = \nu(k)$ and $b = \nu(m,n,k)$, we have

$$q^{-t(m)-t(n^{\vee})}\Delta(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) = \tilde{d}(V)_{a}^{b}(s,l,w,\phi,\rho,t_{\Delta}(m,n,k),x) >_{q} 0. \tag{8.3.3}$$

In particular, $\mathscr{F}$ is $\succ$-merged-log-concave.

Proof. Lemma 8.27 gives the equation in (8.3.3), since $(f_{s,m}f_{s,n^{\vee}})^{-1}f_{s,m\boxminus k}f_{s,(n\boxplus k)^{\vee}} = q^{t_{\Delta}(m,n,k)}$.

Let us prove the inequality in (8.3.3). The monomial conditions of υ give $e(t,m,n,k) \in \mathbb{Z}_{\geq 0}^{l}$ such that $e \leq N_{a}^{b}(2l,w^{\sqcup})$ and $\sum e(t,m,n,k) = t_{\Delta}(m,n,k)$. Then, since ϕ is proper, the inequality in (8.3.3) holds by

$$\tilde{d}(V)_{a}^{b}(s,l,w,\phi,\rho,t_{\Delta}(m,n,k),x,\mathfrak{X}) = d(V)_{a}^{b}(s,l,w,\phi,\rho,e(t,m,n,k),x,\mathfrak{X}) >_{q} 0.$$

We have the merged-log-concavity of $\mathscr{F}$ by the inequality in (8.3.3) and the half $>_{q}$-$\succ$ implication, because $f_{s,m}f_{s,n^{\vee}} = q^{t(m)+t(n^{\vee})} \succ 0$ by the $\succ$-positivity of f_s. □

If there is a $\succ$-admissible $x \in \mathbb{Q}(\mathfrak{X})^{l}$ for some $l \in \mathbb{Z}_{\geq 1}$, then Claim 2 of Lemma 5.20 implies $1 \succ 0$ by $1 >_{x} 0$. Hence, we introduce the following notion of constant parcels.

Definition 8.29. *Suppose a gate $s \geq 0$ and $l \in \mathbb{Z}_{\geq 1}$. Let $1_{s,l} = \left(1_{s,l,m} \in \mathbb{Q}(\mathfrak{X})\right)_{m \in \mathbb{Z}^{l}}$ such that*

$$1_{s,l,m} = \begin{cases} 1 & \text{if } m \in [\![s]\!]^{l}, \\ 0 & \text{otherwise}. \end{cases}$$

Then, we define a constant parcel $\Lambda(s,l,w,\succ,r1_{s,l},\rho,\phi,x,\mathfrak{X})$ when $r \in \mathbb{Q}$ satisfies $r \succ 0$.

These constant parcels yield merged-log-concave parcels for arbitrary gates, widths, positive weights, and base shifts as follows.

Corollary 8.30. *Consider a constant parcel $\mathscr{F} = \Lambda(s,l,w,\succ,r1_{s,l},\rho,\phi,x,\mathfrak{X})$ with $w \in \mathbb{Z}^l_{\geq 1}$ and a proper mediator ϕ. Then, $\mathscr{F}$ is $\succ$-merge-log-concave.*

Proof. The assertion follows from Theorem 8.28 by the functional monomial indices of zero cores. □

In particular, Claim 1 of Proposition 8.26 and Corollary 8.30 give the following explicit merged-log-concave constant parcels.

Example 8.31. If $\mathscr{F} = \Lambda(s,l,w,\succ,1_{s,l},\rho,x,\mathfrak{X})$ with $w \in \mathbb{Z}^l_{\geq 1}$, then

$$\mathscr{F}_m = \begin{cases} \dfrac{1}{(m)^w_x} \text{ for } m \in [\![s]\!]^l, \\ 0 \text{ otherwise.} \end{cases}$$

8.4 Monomial indices and functional monomial indices

We realize monomial indices (l,w,γ) as functional monomial indices (s,l,w,t) of infinite gates s. This gives more explicit merged-log-concave parcels via Theorem 8.28, since $3l$ rational numbers determine the core γ of a monomial index (l,w,γ). Furthermore, by Proposition 7.21, cut operators turn merged-log-concave parcels of infinite gates into merged-log-concave parcels of any gates.

We state the following lemma on fitting tuples.

Lemma 8.32. *Consider an infinite gate $s \geq 0$ and $l \in \mathbb{Z}_{\geq 1}$. For $k \in \mathbb{Z}^{2l}$ and $m \in \mathbb{Z}^l$, let $\mu_{m,k} = (s,l,m,m,k)$.*

1. *Suppose $k \in \mathbb{Z}^{2l}_{\geq 1}$ and a flat $m \in \mathbb{Z}^l$ with $a = \nu(k)$ and $b = \nu(m,m,k)$. Then, $b_j - b_i = a_j - a_i > 0$ for each $1 \leq i < j \leq 2l$.*
2. *Suppose $k \in \mathbb{Z}^{2l}_{\geq 1}$ and a flat $m \in \mathbb{Z}^l$.*
   (a) *If $m \in [\![s]\!]^l$, then $\mu_{m,k}$ is fitting.*
   (b) *If $m \geq \sigma(k)_1 + s_1$, then $\mu_{m,k}$ is wrapped and fitting.*
3. *Let $r \in \mathbb{Z}^l_{\geq 1}$ and $\lambda \in \mathbb{Z}^l$. Then, we have $k \in \mathbb{Z}^{2l}_{\geq 1}$ with the following properties.*
   (a) *For each $i \in [\![l]\!]$, $0 < \sigma(k)_i \equiv \lambda_i \pmod{r_i}$.*
   (b) *$\mu_{m,k}$ is wrapped and fitting for each flat $m \in \mathbb{Z}^l$ such that $m \geq \sigma(k)_1 + s_1$.*
4. *Let $\lambda \in [\![l]\!]$ and $R \in \mathbb{Z}_{\geq 1}$. Then, we have $k \in \mathbb{Z}^{2l}_{\geq 1}$ with the following properties.*
   (a) *$\sigma(k)_i = 2(l-i)+1$ for $i \in [\![\lambda+1,l]\!]$.*
   (b) *$\sigma(k)_i = R+2(l-i)$ for $i \in [\![\lambda]\!]$.*
   (c) *$\mu_{m,k}$ is wrapped and fitting for each flat $m \in \mathbb{Z}^l$ such that $m \geq \sigma(k)_1 + s_1$.*

Proof. Proof of Claim 1. Claim 1 holds, since $b_j - b_i = (a_j + m_1) - (a_i + m_1) = a_j - a_i$ by the flat m and $a_j - a_i = \sum k[i+1:j] > 0$ by $k \in \mathbb{Z}^{2l}_{\geq 1}$.

Proof of Claim 2. Claim 1 gives Claim 2a by $a_1 = k_1 \geq 0$. Claim 2b holds as follows. First, $\mu_{m,k}$ is fitting by Claim 2a, because $m \in [\![s]\!]^l$ by $\sigma(k)_1 \geq 0$ and $s_2 = \infty$. Second, $\mu_{m,k}$ is wrapped, because $m \boxplus k \in [\![s]\!]^l$ and $(m \boxminus k)_i = m_i - \sigma(k)_i \geq m_i - \sigma(k)_1 \geq s_1$ by Lemma 3.9.

Proof of Claim 3. Let us obtain $k \in \mathbb{Z}^{2l}_{\geq 1}$ with Property 3a. Let $k[1:l] = \iota^l(1)$ and $k_{l+1} = p_1 r_l + \lambda_l \in \mathbb{Z}_{\geq 1}$ for some $p_1 \in \mathbb{Z}_{\geq 1}$. Furthermore, we inductively put some $p_i \in \mathbb{Z}_{\geq 1}$ for $i \in [\![2,l]\!]$ so that

$$k_{l+i} = p_i r_{l-i+1} + \lambda_{l-i+1} - \sum k[l-i+2 : l+i-1] \in \mathbb{Z}_{\geq 1}. \tag{8.4.1}$$

This is possible, since $i \in [\![2,l]\!]$ implies $2 \leq l-i+2 < l+i-1 < l+i \leq 2l$. It follows that $i = l$ gives $\sigma(k)_l = k_{l+1} \equiv \lambda_l \pmod{r_l}$. Now, let $i \in [\![l-1]\!]$. Then,

$$2 \leq l-i+1 \leq l. \tag{8.4.2}$$

This gives $2 \leq l-(l-i+1)+2 < l+(l-i+1)-1 \leq 2l-1$ and $(i+1)+1 \leq 2l-(i+1)+1$. Hence, we have

$$\begin{aligned}\sum k[l-(l-i+1)+2 : l+(l-i+1)-1] &= \sum k[i+1 : 2l-i] \\ &= \sum k[(i+1)+1 : 2l-(i+1)+1)] + k_{i+1} \\ &= \sigma(k)_{i+1} + k_{i+1}.\end{aligned}$$

Therefore, since $l-(l-i+1)+1 = i$, the equation in (8.4.1) and inequality (8.4.2) imply

$$\begin{aligned}\sigma(k)_i &= \sum k[i+1 : 2l-i+1] \\ &= \sigma(k)_{i+1} + k_{i+1} + k_{l+l-i+1} \\ &= \sigma(k)_{i+1} + k_{i+1} + p_{l-i+1} r_i + \lambda_i - \sigma(k)_{i+1} - k_{i+1} \\ &\equiv \lambda_i \pmod{r_i}.\end{aligned}$$

Then, Property 3a holds, since $k \in \mathbb{Z}^{2l}_{\geq 1}$ implies $\sigma(k) > 0$. Also, Claim 2b gives a flat $m \in \mathbb{Z}^l$ in Property 3b.

Proof of Claim 4. Let $k \in \mathbb{Z}^{2l}_{\geq 1}$ such that $k_i = 1$ for $2 \leq i \neq \lambda+1 \leq 2l$ and $k_{\lambda+1} = R$. Then, Property 4a holds, since $i \in [\![\lambda+1, l]\!]$ gives

$$\sigma(k)_i = \sum k[i+1 : 2l-i+1] = 2(l-i)+1$$

by $\lambda+1 < i+1 \leq 2l-i+1$. We have Property 4b, because

$$\sigma(k)_i = \sum k[i+1 : 2l-i+1] = (2l-i+1)-i-1+R = R+2(l-i)$$

by $i+1 \leq \lambda+1 \leq 2l-i+1$ for $i \in [\![\lambda]\!]$. Also, Claim 2b gives a flat $m \in \mathbb{Z}^l$ in Property 4c. □

Let us rewrite $n^b_a(2l,w)$.

Lemma 8.33. *Let $l \in \mathbb{Z}_{\geq 1}$. Consider $m,n \in \mathbb{Z}^l$ and $k \in \mathbb{Z}^{2l}$ with $a = \nu(k)$ and $b = \nu(m,n,k)$. Then, $n^b_a(2l,w) = \sum_{i \in [\![l]\!]} w_i \sigma(k)_i (n_{l-i+1} + \sigma(k)_i - m_i)$.*

Proof. The statement holds by Lemma 3.8, since $n^b_a(2l,w) = \sum N^b_a(2l,w)$ for $N^b_a(2l,w)_i = w_i(b_{2l-i+1} - b_i)(a_{2l-i+1} - a_i)$. □

This gives the σ-differences $t_{\gamma,\Delta}$ below by quadratic polynomials t_γ in Definition 1.11.

Lemma 8.34. *Let $l \in \mathbb{Z}_{\geq 1}$ and $\gamma \in \prod_{i \in [\![l]\!]} \mathbb{Q}^3$. Suppose $m,n \in \mathbb{Z}^l$ and $k \in \mathbb{Z}^{2l}$. Then, we have the following.*

1. $t_{\gamma,\Delta}(m,n,k) = 2\sum_{i \in [\![l]\!]} \gamma_{i,1} \sigma(k)_i (n_{l-i+1} + \sigma(k)_i - m_i)$.

2. *If each* $2\gamma_{i,1} \in \mathbb{Z}$*, then* $t_{\gamma,\Delta}(m,n,k) \in \mathbb{Z}$.

Proof. Proof of Claim 1. We have

$$\begin{aligned} t_{\gamma,\Delta}(m,n,k) &= t_\gamma(m \boxminus k) + t_\gamma((n \boxplus k)^\vee) - t_\gamma(m) - t_\gamma(n^\vee) \\ &= \sum_{i\in[\![l]\!]} t_{\gamma,i}(m_i - \sigma(k)_i) + t_{\gamma,i}(n_{l-i+1} + \sigma(k)_i) - t_{\gamma,i}(m_i) - t_{\gamma,i}(n_{l-i+1}) \\ &= \sum_{i\in[\![l]\!],j\in[\![3]\!]} \gamma_{i,j}(m_i - \sigma(k)_i)^{3-j} + \gamma_{i,j}(n_{l-i+1} + \sigma(k)_i)^{3-j} - \gamma_{i,j}m_i^{3-j} - \gamma_{i,j}n_{l-i+1}^{3-j} \\ &= \sum_{i\in[\![l]\!]} \gamma_{i,1}(-2m_i\sigma(k)_i + \sigma(k)_i^2) + \gamma_{i,1}(2n_{l-i+1}\sigma(k)_i + \sigma(k)_i^2) \end{aligned}$$

Proof of Claim 2. Claim 2 follows from Claim 1. □

We discuss the sum monomial condition of monomial indices by the following.

Lemma 8.35. *Let* $l \in \mathbb{Z}_{\geq 1}$ *and* $u, w \in \mathbb{Q}^l$*. Then, each* $h \in [\![l]\!]$ *satisfies*

$$0 \leq \sum_{i\in[\![h]\!]} u_i \leq \sum_{i\in[\![h]\!]} w_i \tag{8.4.3}$$

if and only if each decreasing $\kappa \in \mathbb{Q}^l_{\geq 0}$ *satisfies*

$$0 \leq \sum_{i\in[\![l]\!]} \kappa_i u_i \leq \sum_{i\in[\![l]\!]} \kappa_i w_i.$$

Proof. The if part holds by $\kappa = \iota^h(1) + \iota^{l-h}(0)$. We prove the only if part. When $l = 1$, it holds by $0 \leq u_1 \leq w_1$ and $k_1 \geq 0$. Assume $l \geq 2$.

First, we prove $0 \leq \sum_{i\in[\![l]\!]} \kappa_i u_i$. The left-hand side of inequality (8.4.3) implies $-u_l \leq \sum_{i\in[\![l-1]\!]} u_i$, and hence $\kappa_l u_l \geq -\sum_{i\in[\![l-1]\!]} \kappa_l u_i$ by $\kappa_l \in \mathbb{Q}_{\geq 0}$. The induction on l gives

$$\sum_{i\in[\![l]\!]} \kappa_i u_i = \sum_{i\in[\![l-1]\!]} \kappa_i u_i + \kappa_l u_l \geq \sum_{i\in[\![l-1]\!]} \kappa_i u_i - \sum_{i\in[\![l-1]\!]} \kappa_l u_i = \sum_{i\in[\![l-1]\!]} (\kappa_i - \kappa_l) u_i \geq 0,$$

since $\kappa[1 : l-1] - \kappa_l \in \mathbb{Q}^l_{\geq 0}$ is decreasing.

Second, we obtain $0 \leq \sum_{i\in[\![l]\!]} \kappa_i (w_i - u_i)$ as above, since

$$0 \leq \sum_{i\in[\![h]\!]} (w_i - u_i) \leq \sum_{i\in[\![h]\!]} (w_i - u_i)$$

for each $h \in [\![l]\!]$ by the right-hand side of inequality (8.4.3). □

We now obtain the following equivalence on monomial indices and functional monomial indices by polynomials.

Theorem 8.36. *Suppose an infinite gate* $s \geq 0$, $l \in \mathbb{Z}_{\geq 1}$*, and* $w \in \mathbb{Z}^l_{\geq 1}$*. Consider* $t_i(z) \in \mathbb{Q}[z]$ *for each* $i \in [\![l]\!]$*. Also, let* $t(m) = \sum_{i\in[\![l]\!]} t_i(m_i) \in \mathbb{Q}$ *for each* $m \in \mathbb{Z}^l$*. Then, the following statements are equivalent.*

1. *There is a functional monomial index* $\phi = (s, l, w, t)$*.*

2. *There is a monomial index* $\psi = (l, w, \gamma)$ *such that* $t = t_\gamma$*.*

Proof. Let us prove Statement 1 by Statement 2. First, the integer monomial condition of ϕ follows from that of ψ by Claim 2 of Lemma 8.34. Second, the sum monomial condition of ϕ holds as follows. Suppose a fitting (s,l,m,n,k). Then, Lemma 3.9 gives the decreasing $(\sigma(k)_i(n_{l-i+1}+\sigma(k)_i-m_i))_{i\in[\![l]\!]}\in\mathbb{Z}^l_{\geq 0}$. Then, by Lemma 8.35, the sum monomial condition of ψ gives

$$0\leq 2\sum_{i\in[\![l]\!]}\gamma_{i,1}\sigma(k)_i(n_{l-i+1}+\sigma(k)_i-m_i)\leq\sum_{i\in[\![l]\!]}w_i\sigma(k)_i(n_{l-i+1}+\sigma(k)_i-m_i).$$

We obtain the sum monomial condition of ϕ by Lemma 8.33 and Claim 1 of Lemma 8.34.

Let us prove Statement 2 by Statement 1. First, we prove that each $\deg_z t_i(z)\leq 2$ by contradiction. Assume $d=\max(\deg_z t_1(z),\dots,\deg_z t_l(z))>2$. Suppose all integers $j_1<\dots<j_u$ such that $t_{j_i}(z)=\alpha_{j_i,d}z^d+\alpha_{j_i,d-1}z^{d-1}+\dots$ and $\alpha_{j_i,d}\neq 0$. Also, for a flat $m\in\mathbb{Z}^l$, let

$$\begin{aligned}t_{j_i,\Delta}(m,m,k)&=t_{j_i}((m\boxminus k)_{j_i})+t_{j_i}((m\boxplus k)^{\vee}_{j_i})-t_{j_i}(m_{j_i})-t_{j_i}(m^{\vee}_{j_i})\\&=t_{j_i}(m_1-\sigma(k)_{j_i})+t_{j_i}(m_1+\sigma(k)_{j_i})-t_{j_i}(m_1)-t_{j_i}(m_1).\end{aligned}$$

We deduce $\deg_{m_1}t_{j_i,\Delta}(m,m,k)\leq d-2$, since for each $\lambda\in[\![0,d-1]\!]$, both $m_1^{d-\lambda}$ and $m_1^{d-\lambda-1}$ vanish in

$$\alpha_{j_i,d-\lambda}\left((m_1-\sigma(k)_{j_i})^{d-\lambda}+(m_1+\sigma(k)_{j_i})^{d-\lambda}-m_1^{d-\lambda}-m_1^{d-\lambda}\right).$$

Furthermore, we have

$$t_\Delta(m,m,k)=2m_1^{d-2}\binom{d}{d-2}\sum_{i\in[\![u]\!]}\alpha_{j_i,d}\sigma(k)^2_{j_i}+\dots.\tag{8.4.4}$$

If $\lambda\in[\![l]\!]$ and $R\in\mathbb{Z}_{\geq 1}$, then Claim 4 of Lemma 8.32 gives $h(R,\lambda)\in\mathbb{Z}^{2l}_{\geq 1}$ and a wrapped fitting $\mu(R,\lambda)=(s,l,g(R,\lambda),g(R,\lambda),h(R,\lambda))$ for each flat $g(R,\lambda)\geq\sigma(h(R,\lambda))+s_1$ such that

$$\sigma(h(R,\lambda))_i=\begin{cases}R+2(l-i)\text{ for }i\in[\![\lambda]\!],\\2(l-i)+1\text{ for }i\in[\![\lambda+1,l]\!].\end{cases}$$

For a large enough $R\in\mathbb{Z}_{\geq 1}$, we deduce

$$\sum_{i\in[\![u]\!]}\alpha_{j_i,d}\sigma(h(R,1))^2_{j_i}\neq 0.\tag{8.4.5}$$

Furthermore, for $a(R,\lambda)=\nu(h(R,\lambda))$ and $b(g(R,\lambda),R)=\nu(g(R,\lambda),g(R,\lambda),h(R,\lambda))$, Lemma 8.33 gives

$$n^{b(g(R,\lambda),R)}_{a(R,\lambda)}(2l,w)=\sum_{i\in[\![l]\!]}w_i\sigma(h(R,\lambda))^2_i,\tag{8.4.6}$$

which is independent of $g(R,\lambda)$.

Therefore, since $d>2$, equations (8.4.4) and (8.4.6) and inequality (8.4.5) imply that large $R\in\mathbb{Z}_{\geq 1}$ and $g(R,1)\in\mathbb{Z}^l$ violate the sum monomial condition

$$0\leq t_\Delta(g(R,1),g(R,1),h(R,1))\leq n^{b(g(R,1),R)}_{a(R,1)}(2l,w)$$

of ϕ. This gives $\gamma\in\prod_{i\in[\![l]\!]}\mathbb{Q}^3$ such that $t=t_\gamma$.

Second, we prove the integer monomial condition of ψ by contradiction. Suppose some $j \in [\![l]\!]$ such that $2\gamma_{j,1} \notin \mathbb{Z}$. For each $i \in [\![l]\!]$, let $y_i \geq 1$ and $2\gamma_{i,1} = \frac{x_i}{y_i}$ with coprime x_i and y_i. By Claim 3 of Lemma 8.32, $y = (y_i)_{i \in [\![l]\!]}$ gives a flat $g(y) \in \mathbb{Z}^l$ and wrapped fitting $(s, l, g(y), g(y), h(y))$ such that $\sigma(h(y))_j \equiv 1 \pmod{y_j}$ and $\sigma(h(y))_i \equiv 0 \pmod{y_i}$ if $i \neq j$. Then, we have

$$2\gamma_{j,1}\sigma(h(y))_j^2 \notin \mathbb{Z},$$
$$2\gamma_{i,1}\sigma(h(y))_i^2 \in \mathbb{Z} \text{ if } i \neq j.$$

However, this contradicts the integer monomial condition of ϕ, since $t_\Delta(g(y), g(y), h(y)) = \sum_i 2\gamma_{i,1}\sigma(h(y))_i^2$ by Claim 1 of Lemma 8.34.

Third, we prove the sum monomial condition of ψ. For $\mu(R,\lambda)$, $a(R,\lambda)$, and $b(g(R,\lambda),\lambda)$ above, Lemma 8.33 and Claim 1 of Lemma 8.34 yield

$$\begin{aligned} n_{a(R,\lambda)}^{b(g(R,\lambda),\lambda)}(2l,w) - t_\Delta(g(R,\lambda), g(R,\lambda), w) = &\sum_{i \in [\![\lambda]\!]} (w_i - 2\gamma_{i,1})(R + 2(l-i))^2 \\ &+ \sum_{i \in [\![\lambda+1, l]\!]} (w_i - 2\gamma_{i,1})(2(l-i)+1)^2. \end{aligned}$$

Since $w > 0$, $\sum_{i \in [\![\lambda]\!]} w_i \sigma(h(R,\lambda))_i^2 = \sum_{i \in [\![\lambda]\!]} w_i (R + 2(l-i))^2 > 0$. It follows that

$$\lim_{R \to \infty} \frac{n_{a(R,\lambda)}^{b(g(R,\lambda),\lambda)}(2l,w) - t_\Delta(g(R,\lambda), g(R,\lambda), w)}{\sum_{i \in [\![\lambda]\!]} w_i \sigma(h(R,\lambda))_i^2} = 1 - \frac{\sum_{i \in [\![\lambda]\!]} 2\gamma_{i,1}}{\sum_{i \in [\![\lambda]\!]} w_i}.$$

This limit has to be non-negative by $n_{a(R,\lambda)}^{b(g(R,\lambda),\lambda)}(2l,w) - t_\Delta(g(R,\lambda), g(R,\lambda), w) \geq 0$ in the sum monomial condition of ϕ. We derive

$$\sum_{i \in [\![\lambda]\!]} 2\gamma_{i,1} \leq \sum_{i \in [\![\lambda]\!]} w_i. \tag{8.4.7}$$

Similarly, $\lim_{R \to \infty} \frac{t_\Delta(g(R,\lambda), g(R,\lambda), w)}{\sum_{i \in [\![\lambda]\!]} w_i \sigma(h(R,\lambda))_i^2} = \frac{\sum_{i \in [\![\lambda]\!]} 2\gamma_{i,1}}{\sum_{i \in [\![\lambda]\!]} w_i} \geq 0$ by $t_\Delta(g(R,\lambda), g(R,\lambda), w) \geq 0$ in the sum monomial condition of ϕ. We derive

$$0 \leq \sum_{i \in [\![\lambda]\!]} 2\gamma_{i,1}. \tag{8.4.8}$$

Therefore, the sum monomial condition of ψ follows, since we have inequalities (8.4.7) and (8.4.8) for each $\lambda \in [\![l]\!]$. □

Remark 8.37. The proof above holds, since the flips $n^\vee$ and $(n \boxplus k)^\vee$ in $\Delta(\mathscr{F})(s, l, w, m, n, k, \phi, \rho, x, \mathfrak{X})$ kill $\gamma_{i,2}$ in $t_{\gamma,\Delta}(m,n,k)$ by Claim 1 of Lemma 8.34.

Remark 8.38. Suppose the notation in Theorem 8.36 with $l = 1$. If ϕ is a functional monomial index, then Theorem 8.36 implies $t = t_\gamma$, and hence $\gamma_{1,1} \geq 0$ by the sum monomial condition of ψ. Proposition 13.17 also gives this inequality in some general setting, not necessarily of monomial indices.

8.5 Monomial parcels

By monomial indices, we introduce the notion of monomial parcels to explicitly obtain more merged-log-concave parcels.

Definition 8.39. *Suppose an infinite gate $s \geq 0$ and monomial index (l,w,γ). Let $q \in \mathbb{Q}(\mathfrak{X})$.*

1. *We define the t-monomials $\Psi_{s,\gamma,q} = \left(\Psi_{s,\gamma,q,m} \in \mathbb{Q}(\mathfrak{X})\right)_{m \in \mathbb{Z}^l}$ such that*

$$\Psi_{s,\gamma,q,m} = \begin{cases} q^{t_\gamma(m)} \text{ if } m \in [\![s]\!]^l, \\ 0 \text{ otherwise}. \end{cases}$$

2. *Let q be $\succ$-admissible. For each $m \in [\![s]\!]^l$, assume*

$$\Psi_{s,\gamma,q,m} = q^{t_\gamma(m)} \succ 0.$$

   Suppose a proper $(s,l,w,>_q,\rho,x,\mathfrak{X})$-mediator ϕ for $x = \iota^l(q)$. Then, we define the monomial parcel

$$\mathscr{F} = \Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\phi,\rho,x,\mathfrak{X}).$$

   In particular, we call $\mathscr{F}$ $\frac{q^{t_\gamma(\iota^l(1))}}{q^{t_\gamma(\iota^l(0))}}$-linear (or linear for simplicity) if $(\gamma_{i,1})_{i \in [\![l]\!]} = \iota^l(0)$. Also, we call $\mathscr{F}$ $\frac{q^{t_\gamma(\iota^l(1))}}{q^{t_\gamma(\iota^l(0))}}$-quadratic (or quadratic for simplicity) if $(\gamma_{i,1})_{i \in [\![l]\!]} \neq \iota^l(0)$.

Explicitly, each $m \in [\![s]\!]^l$ gives

$$\mathscr{F}_m = \frac{\Psi_{s,\gamma,q,m}}{\prod \phi(x)^{m \circ w} \cdot [m]!_q^w} = \frac{q^{t_\gamma(m)}}{\prod \phi(x)^{m \circ w} \cdot [m]!_q^w},$$

which is $\frac{q^{t_\gamma(m)}}{(m)_q^w}$ for the canonical mediator ϕ. We now obtain the following q-polynomials with positive integer coefficients.

Theorem 8.40. *Consider a monomial parcel $\mathscr{F} = \Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\phi,\rho,x,\mathfrak{X})$. For each fitting (s,l,m,n,k) with $a = \nu(k)$ and $b = \nu(m,n,k)$, we have*

$$q^{-t_\gamma(m)-t_\gamma(n^\vee)}\Delta(\mathscr{F})(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) = \tilde{d}(V)_a^b(s,l,w,\phi,\rho,t_{\gamma,\Delta}(m,n,k),x) >_q 0.$$

In particular, $\mathscr{F}$ is $\succ$-merged-log-concave.

Proof. Statements follow from Theorem 8.28 and Theorem 8.36. □

If $l = 1$ and $w = (1)$, then the monomial conditions of (l,w,γ) imply $\gamma_{1,1} = 0$ or $\gamma_{1,1} = \frac{1}{2}$. Hence, suppose $\gamma_1 = ((0,0,0))$ and $\gamma_2 = \left(\left(\frac{1}{2},-\frac{1}{2},0\right)\right)$ with $s = (0,\infty)$ so that we have the linear and quadratic $\mathscr{F}_i = \Lambda(s,l,w,\succ,\Psi_{s,\gamma_i,q},\rho,x,\mathfrak{X})$ of $i \in [\![2]\!]$. Then, for an indeterminate t, $(\pm t;q)_\infty^{\mp 1}$ are $\sum_{m \in \mathbb{Z}^l_{\geq 1}} \mathscr{F}_{i,m} t^{m_1}$ of $i \in [\![2]\!]$ by the Euler binomial identities.

Example 8.41. For $s=(0,\infty)$, $l=1$, $w=(1)$, and $\gamma=((0,0,0))$, consider $\mathscr{F}=\Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},x,\mathfrak{X})$. Let $m=n=(2)$ and $k=(1,1)$. Then, we have $a=\nu(k)=(1,2)$, $b=\nu(m,n,k)=(3,4)$, $m\boxminus k=(1)$, $n\boxplus k=(3)$. Hence, we obtain the following q-polynomial with positive integer coefficients:

$$\begin{aligned}\Delta(\mathscr{F})(s,l,w,m,n,k,x,\mathfrak{X}) &= \frac{(3)_q(4)_q}{(1)_q(2)_q}\left(\frac{1}{(2)_q}\cdot\frac{1}{(2)_q}-\frac{1}{(1)_q}\cdot\frac{1}{(3)_q}\right)\\ &= \begin{bmatrix}3\\1\end{bmatrix}_q\begin{bmatrix}4\\2\end{bmatrix}_q-\begin{bmatrix}3\\2\end{bmatrix}_q\begin{bmatrix}4\\1\end{bmatrix}_q\\ &= q^6+q^5+2q^4+q^3+q^2.\end{aligned}$$

Example 8.42. Let $s=(0,\infty)$, $l=2$, $w=\iota^l(1)$, and $\gamma=\iota^l((0,0,0))$. Suppose $\mathscr{F}=\Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},x,\mathfrak{X})$. Also, let $m=n=\iota^l(3)$ and $k=\iota^{2l}(1)$ so that $a=\nu(k)=(1,2,3,4)$, $b=\nu(m,n,k)=(4,5,6,7)$, $m\boxminus k=(0,2)$, $n\boxplus k=(4,6)$. Then, we have the following q-polynomial with positive integer coefficients:

$$\begin{aligned}\Delta(\mathscr{F})(s,l,w,m,n,k,x,\mathfrak{X}) &= \prod_{i\in[\![4]\!]}\frac{(i+3)_q}{(i)_q}\cdot\left(\frac{1}{(3)_q(3)_q}\cdot\frac{1}{(3)_q(3)_q}-\frac{1}{(0)_q(2)_q}\cdot\frac{1}{(4)_q(6)_q}\right)\\ &= \begin{bmatrix}4\\1\end{bmatrix}_q\begin{bmatrix}5\\2\end{bmatrix}_q\begin{bmatrix}6\\3\end{bmatrix}_q\begin{bmatrix}7\\4\end{bmatrix}_q-\begin{bmatrix}4\\4\end{bmatrix}_q\begin{bmatrix}5\\3\end{bmatrix}_q\begin{bmatrix}6\\2\end{bmatrix}_q\begin{bmatrix}7\\1\end{bmatrix}_q\\ &= q^{30}+4q^{29}+13q^{28}+34q^{27}+76q^{26}+151q^{25}+273q^{24}\\ &\quad+452q^{23}+695q^{22}+999q^{21}+1346q^{20}+1710q^{19}+2052q^{18}+2330q^{17}\\ &\quad+2506q^{16}+2557q^{15}+2470q^{14}+2262q^{13}+1958q^{12}+1600q^{11}+1229q^{10}\\ &\quad+886q^{9}+593q^{8}+368q^{7}+208q^{6}+106q^{5}+47q^{4}+18q^{3}+5q^{2}+q.\end{aligned}$$

8.6 On the gap-free property of merged determinants

We adopt the following notation to discuss merged determinants of some monomial parcels.

Definition 8.43. *Let $f\in\mathbb{Q}[q]$.*

1. *We call f q-gap-free if $f_i\neq 0$ for each $i\in\mathbb{Z}$ such that $\mathrm{ord}_q(f)\leq i\leq \deg_q(f)$.*
2. *We write $f>_{q,d} g$ for $g\in\mathbb{Q}[q]$ if $f-g$ is q-gap-free and $f-g>_q 0$.*

We consider the change of variable $q\mapsto q^\rho$ by the gap-free property. Hence, we state the following transitivity of the gap-free property by the base shift function $b_{\lambda,\rho}(q)$.

Lemma 8.44. *Let $\rho,\lambda\in\mathbb{Z}_{\geq 1}$. If $f>_{q^\rho,d}0$, then $fb_{\lambda,\rho}(q)>_{q,d}0$.*

Proof. If $\rho=1$, then the statement follows from $b_{\lambda,\rho}(q)=1$ in Lemma 8.1. Let $\rho\geq 2$. We assume $\mathrm{ord}_q(f)=0$ for simplicity, replacing f by $q^{-\mathrm{ord}_q(f)}f$. If $\lambda=1$, then the statement holds by $b_{\lambda,\rho}(q)=[\rho]_q$ in Lemma 8.1. Let $\lambda\geq 2$. Then, $fb_{\lambda-1,\rho}(q)>_{q,d}0$ by the induction on λ. Therefore, $fb_{\lambda,\rho}(q)=fb_{\lambda-1,\rho}(q)[\rho]_{q^\lambda}>_{q,d}0$ by $\mathrm{ord}_q(fb_{\lambda-1,\rho}(q))=0$, since $\deg_q(b_{\lambda-1,\rho}(q))=\frac{(\rho-1)\lambda(\lambda-1)}{2}\geq\lambda-1$ by Lemma 8.2. ☐

By the following poring, we discuss the binary relation $>_{q,d}$.

Lemma 8.45. $\mathbb{Q}[q]$ *is a strict $>_{q,d}$-poring.*

Proof. First, the irreflexivity holds, as $0 >_{q,d} 0$ is false. Second, suppose $f_1 >_{q,d} f_2 >_{q,d} f_3$. Then, for $j \in [\![2]\!]$ and $i \in [\![\mathrm{ord}_q(f_j), \deg_q(f_j)]\!]$, we have $\mathrm{ord}_q(f_j) \leq \mathrm{ord}_q(f_{j+1})$, $\deg_q(f_j) \geq \deg_q(f_{j+1})$, and $f_{j,i} > f_{j+1,i}$. The transitivity $f_1 >_{q,d} f_3$ follows, since for $i \in [\![\mathrm{ord}_q(f_1), \deg_q(f_1)]\!]$, we have

$$\begin{aligned} \mathrm{ord}_q(f_1) &\leq \mathrm{ord}_q(f_3), \\ \deg_q(f_1) &\geq \deg_q(f_3), \\ f_{1,i} &> f_{3,i}. \end{aligned}$$

Third, if $f_1 >_{q,d} f_2$ and $f_3 \in \mathbb{Q}[q]$, then the additivity follows from $(f_1 + f_3) - (f_2 + f_3) = f_1 - f_2 >_{q,d} 0$. Finally, let $f_1, f_2 >_{q,d} 0$. When $i \in [\![\mathrm{ord}_q(f_1) + \mathrm{ord}_q(f_2), \deg_q(f_1) + \deg_q(f_2)]\!]$, we have $j_k \in [\![\mathrm{ord}_q(f_k), \deg_q(f_k)]\!]$ for $k \in [\![2]\!]$ such that $j_1 + j_2 = i$. The multiplicativity $f_1 f_2 >_{q,d} 0$ follows from $(f_1 f_2)_i \geq f_{1,j_1} f_{2,j_2} > 0$. ☐

Hence, we have the following gap-free shifted x-binomial products.

Proposition 8.46. *Let $\mu \in \mathbb{Z}_{\geq 1}$ and $x = \iota^{\mu}(q)$. Consider $a, b \in \mathbb{Z}^{\mu}$ such that $b \geq a \geq 0$. Then, we have $U_a^b(\mu, w, \rho, x) >_{q,d} 0$.*

Proof. Since $\begin{bmatrix} b_i \\ a_i \end{bmatrix}_{q^{\rho_i}} >_{q^{\rho_i},d} 0$, we have $U_{(a_i)}^{(b_i)}(1,(1),(\rho_i),(q)) = \begin{bmatrix} b_i \\ a_i \end{bmatrix}_{q^{\rho_i}} b_{b_i - a_i, \rho_i}(q) >_{q,d} 0$ by Lemma 8.44. Hence, Lemma 8.45 gives the assertion, since $U_a^b(\mu, w, \rho, x) = \prod_{i \in [\![\mu]\!]} U_{(a_i)}^{(b_i)}(1,(1),(\rho_i),(q))^{w_i}$. ☐

To discuss pre-merged determinants, we introduce the following notion on the tempered pre-fitting tuples.

Definition 8.47. *Suppose $\upsilon = (\mu, a, b)$ for $\mu \in \mathbb{Z}_{\geq 2}$ and $a, b \in \mathbb{Z}^{\mu}$.*

1. *We call υ strictly pre-fitting if $1 \leq a_1 < \cdots < a_\mu \leq b_1 < \cdots < b_\mu$.*
2. *We call υ almost strictly pre-fitting if $0 \leq a_1 < \cdots < a_\mu \leq b_1 < \cdots < b_\mu$.*

Then, we conjecture the following q-gap-free property of pre-merged determinants. This implies the q-gap-free property of merged determinants of some width-two monomial parcels by Lemma 8.25 and Theorem 8.40.

Conjecture 8.48. *Let $\mu = 2$, $w = \iota^{\mu}(1)$, and $x = \iota^{\mu}(q)$. Suppose a flat $\rho \in \mathbb{Z}_{\geq 1}^{\mu}$ and strictly pre-fitting $\upsilon = (\mu, a, b)$. Let $e \in [\![0, n_a^b(\mu, w)]\!]$. Then,*

$$\tilde{d}(U)_a^b(\mu, w, \rho, e, x) >_{q,d} 0.$$

Example 8.49. Conjecture 8.48 does not extend to almost strictly pre-fitting tuples. For example, if $w = \rho = \iota^2(1)$, $a = (0,2)$, $b = (2,3)$, and $e = 1$, then

$$\tilde{d}(U)_a^b(2, w, \rho, e, x) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}_q \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q - q \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q \begin{bmatrix} 3 \\ 0 \end{bmatrix}_q = q^2 + 1 \not>_{q,d} 0.$$

We obtain the following gap-free pre-merged determinants, assuming Conjecture 8.48 for the width-two cases.

Theorem 8.50. *Assume Conjecture 8.48. Let $\mu \in \mathbb{Z}_{\geq 2}$ and $x = \iota^{\mu}(q)$. Consider a pre-fitting $\kappa = (\mu, a, b)$ such that $a > 0$. Let $e \in [\![0, n_a^b(\mu, w)]\!]$. Then, we have*

$$\tilde{d}(U)_a^b(\mu, w, \rho, e, x) >_{q,d} 0.$$

Proof. If κ is not tempered, then $U^b_{a^\vee}(\mu,w,\rho,x)=0$ implies the assertion by Proposition 8.46. Hence, assume that κ is tempered.

If $\mu=2$, then the assertion becomes Conjecture 8.48. Suppose an odd $\mu\geq 3$. Then, Claim 2 of Lemma 8.8 gives

$$\tilde{d}(U)^b_a(\mu,w,\rho,e,x)=U^{\mathrm{C}(b)}_{\mathrm{C}(a)}(1,\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x))\cdot\tilde{d}(U)^{\mathrm{O}(b)}_{\mathrm{O}(a)}(\mu-1,\mathrm{O}(\rho),\mathrm{O}(w),e,\mathrm{O}(x)).$$

Also, $0\leq e\leq n^b_a(\mu,w)=n^{\mathrm{O}(b)}_{\mathrm{O}(a)}(\mu-1,\mathrm{O}(w))$. Therefore, the assertion holds by Lemma 8.45 and the induction on μ, since $U^{\mathrm{C}(b)}_{\mathrm{C}(a)}(1,\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x))>_{q,d}0$ by Proposition 8.46.

Suppose an even $\mu\geq 2$. Let $\mu=2l$. Consider $E\in\mathbb{Z}^l_{\geq 0}$ such that $E\leq N^b_a(\mu,w)$ and $\sum E=e$. Then, Claim 2 of Lemma 8.8 implies

$$\begin{aligned}&\tilde{d}(U)^b_a(\mu,w,\rho,e,x)\\&=U^{\mathrm{C}(b)}_{\mathrm{C}(a)}(2,\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x))\cdot U^{\mathrm{O}(b)}_{\mathrm{O}(a)}(\mu-2,\mathrm{O}(w),\mathrm{O}(\rho),\mathrm{O}(x))-\\&\prod x[l:l]^{\mathrm{C}(\mu,E)}\cdot U^{\mathrm{C}(b)}_{\mathrm{C}(a)^\vee}(2,\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x))\cdot\prod x[1:l-1]^{\mathrm{O}(\mu,E)}U^{\mathrm{O}(b)}_{\mathrm{O}(a)^\vee}(\mu-2,\mathrm{O}(w),\mathrm{O}(\rho),\mathrm{O}(x)).\end{aligned}\quad(8.6.1)$$

Also, since $\mathrm{O}(a)^\vee\leq\mathrm{O}(b)$ for the tempered κ and $\prod x[1:l-1]^{\mathrm{O}(\mu,E)}$ is a q-monomial, the induction on μ and Proposition 8.46 give

$$U^{\mathrm{O}(b)}_{\mathrm{O}(a)}(\mu-2,\mathrm{O}(w),\mathrm{O}(\rho),\mathrm{O}(x))>_{q,d}\prod x[1:l-1]^{\mathrm{O}(\mu,E)}U^{\mathrm{O}(b)}_{\mathrm{O}(a)^\vee}(\mu-2,\mathrm{O}(w),\mathrm{O}(\rho),\mathrm{O}(x))>_{q,d}0.\quad(8.6.2)$$

First, assume that $\mathrm{C}(a)$ or $\mathrm{C}(b)$ is flat. Then, $N^b_a(\mu,w)_l=w_l(b_{l+1}-b_l)(a_{l+1}-a_l)=0$ gives $\mathrm{C}(\mu,E)=(0)$. Since $\mathrm{C}(b)\geq\mathrm{C}(a)^\vee=\mathrm{C}(a)$ for the tempered κ and flat $\mathrm{C}(a)$, Proposition 8.46 implies

$$\prod x[l:l]^{\mathrm{C}(\mu,E)}\cdot U^{\mathrm{C}(b)}_{\mathrm{C}(a)^\vee}(2,\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x))=U^{\mathrm{C}(b)}_{\mathrm{C}(a)}(2,\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x))>_{q,d}0.$$

Then, equation (8.6.1) and inequality (8.6.2) give the assertion by Claim 2b of Lemma 5.2 and Lemma 8.45.

Second, assume $\mathrm{C}(a)_1<\mathrm{C}(a)_2$ and $\mathrm{C}(b)_1<\mathrm{C}(b)_2$. We have $\mathrm{C}(b)\geq\mathrm{C}(a)^\vee$ for the tempered κ and $\mathrm{C}(\mu,E)\leq N^{\mathrm{C}(b)}_{\mathrm{C}(a)}(2,\mathrm{C}(w))$. The induction on μ and Proposition 8.46 give

$$U^{\mathrm{C}(b)}_{\mathrm{C}(a)}(2,\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x))>_{q,d}\prod x[l:l]^{\mathrm{C}(\mu,E)}U^{\mathrm{C}(b)}_{\mathrm{C}(a)^\vee}(2,\mathrm{C}(w),\mathrm{C}(\rho),\mathrm{C}(x))>_{q,d}0.$$

Again, equation (8.6.1) and inequality (8.6.2) give the assertion by Claim 2d of Lemma 5.2 and Lemma 8.45. □

8.7 On the almost log-concavity, unimodality, and palindromicity of pre-merged determinants

We discuss the log-concavity, unimodality, and palindromicity by shifted x-binomial products and pre-merged determinants, which allow odd widths unlike quasi-merged determinants. We adopt the following terminology to avoid conjecturing upon Conjecture 8.48.

Definition 8.51. *Let $f\in\mathbb{Q}[q]$. Let $u(f)=(f_{\lambda_i})_{i\in[\![d]\!]}$ such that $f=\sum_{i\in[\![d]\!]}f_{\lambda_i}q^{\lambda_i}$ and each $u(f)_i\neq 0$. We call the q-polynomial f almost palindromic if $u(f)$ is palindromic, almost unimodal if $u(f)$ is unimodal, and almost log-concave if $u(f)$ is log-concave.*

8.7.1 On the almost log-concavity

We state the following almost log-concavity on shifted x-binomial products. It is well-known that q-binomial coefficients are unimodal, but not necessarily log-concave.

Conjecture 8.52. *Let $\lambda, \delta \in \mathbb{Z}_{\geq 1}$ and $x = (q)$. Then, there exists $h_{\delta,\lambda} \in \mathbb{Z}_{\geq 1}$ such that*

$$U_{(a_1)}^{(b_1)}(1, (h_{\delta,\lambda}), (t), x) = \begin{bmatrix} b_1 \\ a_1 \end{bmatrix}_{q^t}^{h_{\delta,\lambda}} b_{b_1-a_1,t}(q)^{h_{\delta,\lambda}}$$

is a log-concave q-polynomial for any $t \in [\![\lambda]\!]$ and $1 \leq a_1 < b_1 \leq \delta$.

Example 8.53. The q-polynomial

$$U_{(2)}^{(4)}(1, (3), (1), (q)) \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q^3 = q^{12} + 3q^{11} + 9q^{10} + 16q^9 + 27q^8 + 33q^7 + 38q^6 + 33q^5 + 27q^4 + 16q^3 + 9q^2 + 3q + 1$$

is log-concave, unlike

$$U_{(2)}^{(4)}(1, (1), (1), (q)) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = q^4 + q^3 + 2q^2 + q + 1.$$

One can check that setting $h_{30,20} = 3$ supports Conjecture 8.52.

We then state the following analog on pre-merged determinants.

Conjecture 8.54. *Let $\mu = 2$, $x = \iota^\mu(q)$, and $\delta, \lambda \in \mathbb{Z}_{\geq 1}$. Then, there exists $H_{\delta,\lambda} \in \mathbb{Z}_{\geq 1}$ such that the pre-merged determinant*

$$\tilde{d}(U)_a^b(\mu, \iota^\mu(H_{\delta,\lambda}), \iota^\mu(t), e, x)$$

is an almost log-concave q-polynomial whenever we have a strictly pre-fitting (μ, a, b) with $b \leq \delta$, $e \in [\![0, n_a^b(\mu, \iota^\mu(H_{\delta,\lambda}))]\!]$, and $t \in [\![\lambda]\!]$.

Example 8.55. The q-polynomial

$$\begin{aligned} \tilde{d}(U)_{(1,2)}^{(3,4)}(2, \iota^2(3), \iota^2(1), 0, \iota^2(q)) &= \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q^3 \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q^3 - \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q^3 \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q^3 \\ &= q^{18} + 6q^{17} + 24q^{16} + 67q^{15} + 150q^{14} + 273q^{13} + 422q^{12} + 555q^{11} \\ &\quad + 633q^{10} + 622q^9 + 531q^8 + 387q^7 + 241q^6 + 123q^5 + 51q^4 + 15q^3 + 3q^2 \end{aligned}$$

is log-concave, unlike

$$\tilde{d}(U)_{(1,2)}^{(3,4)}(2, \iota^2(1), \iota^2(1), 0, \iota^2(q)) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q - \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q = q^6 + q^5 + 2q^4 + q^3 + q^2.$$

One can check that setting $H_{20,10} = 3$ supports Conjecture 8.54.

We also conjecture the following on higher-width pre-merged determinants.

Conjecture 8.56. *For $\mu \in \mathbb{Z}_{\geq 4}$ and $\lambda \in \mathbb{Z}_{\geq 0}$, let (μ, a, b) be almost strictly pre-fitting. Then,*

$$\tilde{d}(U)_a^b(\mu, \iota^\mu(1), \iota^\mu(\lambda), h_a^b(\mu, \iota^\mu(1)), \iota^\mu(q))$$

is an almost log-concave q-polynomial.

Example 8.57. We have $h_{(0,2,3)}^{(3,5,7)}(3, \iota^3(1)) = h_{(0,1,2,3)}^{(3,4,5,7)}(4, \iota^4(1)) = 6$. The following q-polynomial

$$\begin{aligned}\tilde{d}(U)_{(0,1,2,3)}^{(3,4,5,7)}(4, \iota^4(1), \iota^4(1), 6, \iota^4(q)) = {} & q^{21} + 3q^{20} + 8q^{19} + 17q^{18} + 31q^{17} + 50q^{16} + 74q^{15} + 98q^{14} \\ & + 121q^{13} + 138q^{12} + 147q^{11} + 146q^{10} + 137q^9 + 119q^8 \\ & + 97q^7 + 73q^6 + 50q^5 + 31q^4 + 17q^3 + 8q^2 + 3q + 1\end{aligned}$$

is log-concave, unlike

$$\begin{aligned}\tilde{d}(U)_{(0,2,3)}^{(3,5,7)}(3, \iota^3(1), \iota^3(1), 6, \iota^3(q)) = {} & q^{18} + 2q^{17} + 5q^{16} + 9q^{15} + 15q^{14} + 21q^{13} + 28q^{12} + 33q^{11} + 37q^{10} \\ & + 38q^9 + 37q^8 + 33q^7 + 28q^6 + 21q^5 + 15q^4 + 9q^3 + 5q^2 + 2q + 1.\end{aligned}$$

8.7.2 On the almost unimodality

Example 8.58. A pre-merged determinant does not have to be almost unimodal, because

$$\tilde{d}(U)_{(0,2)}^{(2,5)}(2, \iota^2(1), \iota^2(1), 3, \iota^2(q)) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}_q \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q - q^3 \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q \begin{bmatrix} 5 \\ 0 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + 2q^4 + q^5 + q^6.$$

Even with the trivial degree shift,

$$\tilde{d}(U)_{(0,6)}^{(11,14)}(2, \iota^2(1), \iota^2(1), 0, \iota^2(q)) = \begin{bmatrix} 11 \\ 0 \end{bmatrix}_q \begin{bmatrix} 14 \\ 6 \end{bmatrix}_q - \begin{bmatrix} 11 \\ 6 \end{bmatrix}_q \begin{bmatrix} 14 \\ 0 \end{bmatrix}_q$$

is not unimodal either, as it is

$$\begin{aligned} & q^{48} + q^{47} + 2q^{46} + 3q^{45} + 5q^{44} + 7q^{43} + 11q^{42} + 14q^{41} + 20q^{40} + 25q^{39} + 33q^{38} + 40q^{37} \\ & + 51q^{36} + 59q^{35} + 71q^{34} + 81q^{33} + 94q^{32} + 103q^{31} + 115q^{30} + 122q^{29} + 132q^{28} + 136q^{27} \\ & + 141q^{26} + 140q^{25} + 141q^{24} + 135q^{23} + 130q^{22} + 120q^{21} + 111q^{20} + 98q^{19} + 87q^{18} \\ & + 73q^{17} + 62q^{16} + 49q^{15} + 39q^{14} + 29q^{13} + 22q^{12} + 15q^{11} + 10q^{10} + 6q^9 + 4q^8 + 2q^7 + q^6. \end{aligned} \tag{8.7.1}$$

However, we conjecture the following for strictly pre-fitting tuples.

Conjecture 8.59. *Let $\mu = 2$. When (μ, a, b) is strictly pre-fitting,*

$$\tilde{d}(U)_a^b(\mu, \iota^\mu(1), \iota^\mu(1), 0, \iota^\mu(q))$$

is an almost unimodal q-polynomial.

Example 8.60. By (8.7.1), the q-polynomial $\tilde{d}(U)_{(0,6)}^{(11,14)}(2, \iota^2(1), \iota^2(1), 0, \iota^2(q))$ is not unimodal. However,

$$\tilde{d}(U)_{(1,6)}^{(11,14)}(2, \iota^2(1), \iota^2(1), 0, \iota^2(q))$$

is unimodal, as it is

$$q^{58}+2q^{57}+4q^{56}+7q^{55}+12q^{54}+19q^{53}+30q^{52}+44q^{51}+64q^{50}+89q^{49}+122q^{48}+161q^{47}$$
$$+211q^{46}+268q^{45}+336q^{44}+411q^{43}+497q^{42}+587q^{41}+686q^{40}+784q^{39}+886q^{38}+982q^{37}$$
$$+1076q^{36}+1156q^{35}+1229q^{34}+1282q^{33}+1322q^{32}+1338q^{31}+1339q^{30}+1315q^{29}+1277q^{28}$$
$$+1216q^{27}+1144q^{26}+1055q^{25}+961q^{24}+856q^{23}+753q^{22}+647q^{21}+548q^{20}+452q^{19}+367q^{18}$$
$$+289q^{17}+224q^{16}+167q^{15}+122q^{14}+85q^{13}+58q^{12}+37q^{11}+23q^{10}+13q^{9}+7q^{8}+3q^{7}+q^{6}.$$

We also conjecture the following on higher-width pre-merged determinants.

Conjecture 8.61. *Let $\mu \in \mathbb{Z}_{\geq 3}$ and $\lambda \in \mathbb{Z}_{\geq 1}$. When (μ, a, b) is almost strictly pre-fitting,*

$$\tilde{d}(U)^{b}_{a}(\mu, \iota^{\mu}(1), \iota^{\mu}(\lambda), 0, \iota^{\mu}(q))$$

is an almost unimodal q-polynomial.

Example 8.62. By (8.7.1), $\tilde{d}(U)^{(11,14)}_{(0,6)}(2, \iota^2(1), \iota^2(1), 0, \iota^2(q))$ is not unimodal. However,

$$\tilde{d}(U)^{(11,12,14)}_{(0,2,6)}(3, \iota^3(1), \iota^3(1), 0, \iota^3(q))$$

is unimodal, as it is

$$q^{68}+2q^{67}+5q^{66}+9q^{65}+17q^{64}+28q^{63}+47q^{62}+72q^{61}+111q^{60}+161q^{59}+233q^{58}$$
$$+322q^{57}+443q^{56}+588q^{55}+775q^{54}+993q^{53}+1262q^{52}+1565q^{51}+1924q^{50}+2315q^{49}$$
$$+2761q^{48}+3230q^{47}+3744q^{46}+4263q^{45}+4809q^{44}+5335q^{43}+5862q^{42}+6338q^{41}$$
$$+6786q^{40}+7153q^{39}+7465q^{38}+7671q^{37}+7802q^{36}+7813q^{35}+7742q^{34}+7552q^{33}$$
$$+7286q^{32}+6917q^{31}+6492q^{30}+5992q^{29}+5465q^{28}+4897q^{27}+4333q^{26}+3762q^{25}$$
$$+3223q^{24}+2705q^{23}+2238q^{22}+1810q^{21}+1441q^{20}+1118q^{19}+853q^{18}+631q^{17}$$
$$+458q^{16}+320q^{15}+219q^{14}+143q^{13}+91q^{12}+54q^{11}+31q^{10}+16q^{9}+8q^{8}+3q^{7}+q^{6}.$$

8.7.3 On the almost palindromicity

Conjecture 8.63. *Let $\mu \in \mathbb{Z}_{\geq 3}$. If (μ, a, b) is almost strictly pre-fitting, then*

$$\tilde{d}(U)^{b}_{a}(\mu, \iota^{\mu}(1), \iota^{\mu}(1), 0, \iota^{\mu}(q))$$

is not an almost palindromic q-polynomial.

In particular, $\tilde{d}(U)^{(b_1,b_2)}_{(a_1,a_2)}(2, \iota^2(1), \iota^2(1), 0, \iota^2(q))$ would not be palindromic when $a_2 - a_1 \geq 2$ and $b_2 - b_1 \geq 2$, since these a_1, a_2, b_1, b_2 give $\mu = 3$ cases in Conjecture 8.63. Hence, if $\mu = 2$, then we conjecture the following, which provides infinitely many almost palindromic unimodal q-polynomials.

Conjecture 8.64. *For $\lambda \in \mathbb{Z}_{\geq 0}$, let*

$$M(\lambda) = \left\{ \mu \in \mathbb{Z}^4 \mid 0 \leq \mu(1) < \mu(2) \leq \mu(3) < \mu(4) = \lambda \right\},$$
$$N(\lambda) = \left\{ \mu \in M(\lambda) \mid \tilde{d}(U)^{\mu(3,4)}_{\mu(1,2)}(2, \iota^2(1), \iota^2(1), 0, \iota^2(q)) \text{ is not almost palindromic.} \right\}.$$

Then, $O(\lambda) = \frac{\#(N(\lambda))}{\#(M(\lambda))}$ satisfies $0 = O(3) < O(4) < O(5) < \dots$.

On base shifts, we have the following width-two pre-merged determinants.

Example 8.65. The pre-merged determinant

$$\tilde{d}(U)^{(1,4)}_{(0,1)}(2,\iota^2(1),\iota^2(1),1,\iota^2(q)) = q^3+q^2+1$$

is palindromic, unlike

$$\begin{aligned}\tilde{d}(U)^{(1,4)}_{(0,1)}(2,\iota^2(1),\iota^2(2),1,\iota^2(q)) &= q^{13}+2q^{12}+2q^{11}+4q^{10}+5q^9+5q^8+6q^7\\ &\quad +6q^6+5q^5+4q^4+4q^3+2q^2+q+1.\end{aligned}$$

However, we conjecture the following palindromicity transitivity on base shifts.

Conjecture 8.66. *For $\mu \in \mathbb{Z}_{\geq 3}$, let (μ,a,b) be almost strictly pre-fitting. Suppose $\lambda \in \mathbb{Z}_{\geq 1}$ and $e \in [\![0, n^b_a(\mu,\iota^\mu(1))]\!]$. Then,*

$$\tilde{d}(U)^b_a(\mu,\iota^\mu(1),\iota^\mu(\lambda),e,\iota^\mu(q))$$

is an almost palindromic q-polynomial if and only if

$$\tilde{d}(U)^b_a(\mu,\iota^\mu(1),\iota^\mu(\lambda+1),e,\iota^\mu(q))$$

is an almost palindromic q-polynomial.

9 Separable products

We introduce separable products on parcels to obtain more merged-log-concave parcels with increased widths. For this, we first define the following truncations of fitting tuples.

Definition 9.1. *Assume a fitting $\mu = (s,l_1,m_1,n_1,k_1)$. Let $\lambda \in \mathrm{T}_{\leq}(2,l_1)$ and $l_2 = \lambda_2 - \lambda_1 + 1$.*

1. *We define the truncation $t(\lambda,\mu) = (s,l_2,m_2,n_2,k_2)$ such that*

$$m_2 = m_1[\lambda_1 : \lambda_2], \tag{9.0.1}$$
$$n_2 = n_1[l_1-\lambda_2+1 : l_1-\lambda_1+1], \tag{9.0.2}$$
$$k_{2,1} = \sum k_1[1:\lambda_1], \tag{9.0.3}$$
$$k_2[2:l_2] = k_1[\lambda_1+1:\lambda_2], \tag{9.0.4}$$
$$k_{2,l_2+1} = \sum k_1[\lambda_2+1 : 2l_1-\lambda_2+1], \tag{9.0.5}$$
$$k_2[l_2+2 : 2l_2] = k_1[2l_1-\lambda_2+2 : 2l_1-\lambda_1+1]. \tag{9.0.6}$$

2. *We define the outer truncation $\mathrm{ot}(l_2,\mu) = t(\lambda,\mu)$ if $\lambda_1 = 1$.*
3. *We define the center truncation $\mathrm{ct}(l_2,\mu) = t(\lambda,\mu)$ if $\lambda_2 = l_1$.*

These truncations have the following properties.

Proposition 9.2. *For a fitting $\mu_1 = (s,l_1,m_1,n_1,k_1)$, suppose*

$$\mu_2 = (s,l_2,m_2,n_2,k_2) = t(\lambda,\mu).$$

Let $a_i = \nu(k_i)$ and $b_i = \nu(m_i,n_i,k_i)$ for $i \in [\![2]\!]$. Then, we have the following.

1. *There exist the following equations:*

$$a_2[1:l_2] = a_1[\lambda_1:\lambda_2]; \tag{9.0.7}$$
$$a_2[l_2+1:2l_2] = a_1[2l_1-\lambda_2+1:2l_1-\lambda_1+1]; \tag{9.0.8}$$
$$b_2[1:l_2] = b_1[\lambda_1:\lambda_2]; \tag{9.0.9}$$
$$b_2[l_2+1:2l_2] = b_1[2l_1-\lambda_2+1:2l_1-\lambda_1+1]. \tag{9.0.10}$$

2. *μ_2 is fitting.*

3. *There exist the following equations:*

$$\sigma(k_2) = \sigma(k_1)[\lambda_1:\lambda_2]; \tag{9.0.11}$$
$$m_2 \boxminus k_2 = (m_1 \boxminus k_1)[\lambda_1:\lambda_2]; \tag{9.0.12}$$
$$(n_2 \boxplus k_2)^\vee = (n_1 \boxplus k_1)^\vee[\lambda_1:\lambda_2]. \tag{9.0.13}$$

4. *If μ_1 is wrapped, then μ_2 is wrapped.*

Proof. Proof of Claim 1. Since $a_{2,1} = \sum k_1[1:\lambda_1] = a_{1,\lambda_1}$ by equation (9.0.3), equation (9.0.4) gives equation (9.0.7). Since $a_{2,l_2+1} = a_{1,2l_1-\lambda_2+1}$ by equation (9.0.5), equation (9.0.6) gives equation (9.0.8). By $b_2 = a_2 + m_2 \mathbin{+\!\!\!+} n_2$, equations (9.0.9) and (9.0.10) follow from equations (9.0.1), (9.0.2), (9.0.7), and (9.0.8).

Proof of Claim 2. First, the inclusion condition of μ_1 imply that of μ_2 by equations (9.0.1) and (9.0.2). Second, the slope conditions of μ_1 imply those those of μ_2 by equations (9.0.7), (9.0.8), (9.0.9), and (9.0.10).

Proof of Claim 3. First, we obtain equation (9.0.11), since by equations (9.0.4), (9.0.5), and (9.0.6), each $i \in [\![l_2]\!]$ satisfies

$$\begin{aligned}
\sigma(k_2)_i &= \sum k_2[i+1:2l_2-i+1] \\
&= \sum k_2[i+1:l_2+(l_2-i+1)] \\
&= \sum k_1[i+\lambda_1:2l_1-\lambda_2+(l_2-i+1)] \\
&= \sum k_1[(i+\lambda_1-1)+1:2l_1-(i+\lambda_1-1)+1] \\
&= \sigma(k_1)_{i+\lambda_1-1}.
\end{aligned}$$

Second, equations (9.0.12) and (9.0.13) hold, because for each $i \in [\![l_2]\!]$, equation (9.0.11) gives

$$\begin{aligned}
(m_2 \boxminus k_2)_i &= m_{2,i} - \sigma(k_2)_i \\
&= m_{1,i+\lambda_1-1} - \sigma(k_1)_{i+\lambda_1-1} \\
&= (m_1 \boxminus k_1)_{i+\lambda_1-1}, \\
(n_2 \boxplus k_2)_i &= n_{2,i} + \sigma(k_2)_{l_2-i+1} \\
&= n_{1,l_1-\lambda_2+i} + \sigma(k_1)_{l_2-i+1+\lambda_1-1} \\
&= n_{1,l_1-\lambda_2+i} + \sigma(k_1)_{l_1-(l_1-\lambda_2+i)+1} \\
&= (n_1 \boxplus k_1)_{l_1-\lambda_2+i}.
\end{aligned}$$

Third, for each $i \in [\![0, l_2 - 1]\!]$, we now obtain

$$\begin{aligned}(n_1 \boxplus k_1)^{\vee}_{\lambda_1+i} &= (n_1 \boxplus k_1)_{l_1-(\lambda_1+i)+1} \\ &= (n_1 \boxplus k_1)_{l_1-\lambda_2+l_2-i} \\ &= (n_2 \boxplus k_2)_{l_2-i} \\ &= (n_2 \boxplus k_2)^{\vee}_{i+1}.\end{aligned}$$

Proof of Claim 4. Claim 4 follows from Claim 3. □

We now have the following for center and outer truncations.

Corollary 9.3. *Let $\kappa \in \mathbb{Z}^3_{\geq 1}$ such that $\kappa_3 = \kappa_1 + \kappa_2$. For a fitting $\mu_3 = (s, \kappa_3, m_3, n_3, k_3)$, consider*

$$\begin{aligned}\mu_1 &= \mathrm{ot}(\kappa_1, \mu_3) = (s, \kappa_1, m_1, n_1, k_1), \\ \mu_2 &= \mathrm{ct}(\kappa_2, \mu_3) = (s, \kappa_2, m_2, n_2, k_2).\end{aligned}$$

Let $a_i = \nu(k_i)$ and $b_i = \nu(m_i, n_i, k_i)$ for each $i \in [\![3]\!]$. Also, suppose $u_i \in \mathbb{Z}^{\kappa_i}_{\geq 0}$ and an indeterminate $x_i \in \mathbb{Q}(\mathfrak{X})^{\kappa_i}$ for each $i \in [\![3]\!]$. Then, we have the following.

1. *μ_1 is fitting.*

2. *There exist the following equations:*

$$m_1 = m_3[1 : \kappa_1]; \tag{9.0.14}$$
$$n_1 = n_3[\kappa_3 - \kappa_1 + 1 : \kappa_3]; \tag{9.0.15}$$
$$a_1 = a_3[1 : \kappa_1] + a_3[2\kappa_3 - \kappa_1 + 1 : 2\kappa_3]; \tag{9.0.16}$$
$$b_1 = b_3[1 : \kappa_1] + b_3[2\kappa_3 - \kappa_1 + 1 : 2\kappa_3]; \tag{9.0.17}$$
$$\sigma(k_1) = \sigma(k_3)[1 : \kappa_1]; \tag{9.0.18}$$
$$m_1 \boxminus k_1 = (m_3 \boxminus k_3)[1 : \kappa_1]; \tag{9.0.19}$$
$$(n_1 \boxplus k_1)^{\vee} = (n_3 \boxplus k_3)^{\vee}[1 : \kappa_1]. \tag{9.0.20}$$

3. *Furthermore, we have*

$$\begin{aligned}\begin{bmatrix} b_1 \\ a_1 \end{bmatrix}^{u_1^{\sqcup}}_{x_1^{\sqcup}} &= \begin{bmatrix} b_3[1 : \kappa_1] \\ a_3[1 : \kappa_1] \end{bmatrix}^{u_1}_{x_1} \begin{bmatrix} b_3[2\kappa_3 - \kappa_1 + 1 : 2\kappa_3] \\ a_3[2\kappa_3 - \kappa_1 + 1 : 2\kappa_3] \end{bmatrix}^{u_1^{\vee}}_{x_1^{\vee}}, \\ \begin{bmatrix} b_1 \\ a_1^{\vee} \end{bmatrix}^{u_1^{\sqcup}}_{x_1^{\sqcup}} &= \begin{bmatrix} b_3[1 : \kappa_1] \\ a_3[2\kappa_3 - \kappa_1 + 1 : 2\kappa_3]^{\vee} \end{bmatrix}^{u_2}_{x_2} \begin{bmatrix} b_3[2\kappa_3 - \kappa_1 + 1 : 2\kappa_3] \\ a_3[1 : \kappa_1]^{\vee} \end{bmatrix}^{u_1^{\vee}}_{x_1^{\vee}}.\end{aligned}$$

4. *μ_2 is fitting.*

5. *There exist the following equations:*

$$m_2 = m_3[\kappa_1 + 1 : \kappa_3];$$
$$n_2 = n_3[1 : \kappa_3 - \kappa_1];$$
$$a_2 = a_3[\kappa_1 + 1 : 2\kappa_3 - \kappa_1];$$
$$b_2 = b_3[\kappa_1 + 1 : 2\kappa_3 - \kappa_1];$$
$$\sigma(k_2) = \sigma(k_3)[\kappa_1 + 1 : \kappa_3];$$
$$m_2 \boxminus k_2 = (m_3 \boxminus k_3)[\kappa_1 + 1 : \kappa_3];$$
$$(n_2 \boxplus k_2)^\vee = (n_3 \boxplus k_3)^\vee[\kappa_1 + 1 : \kappa_3].$$

6. *We have*

$$\begin{bmatrix} b_2 \\ a_2 \end{bmatrix}_{x_2^\sqcup}^{u_2^\sqcup} = \begin{bmatrix} b_3[\kappa_1 + 1 : \kappa_3] \\ a_3[\kappa_1 + 1 : \kappa_3] \end{bmatrix}_{x_2}^{u_2} \begin{bmatrix} b_3[\kappa_3 + 1 : 2\kappa_3 - \kappa_1] \\ a_3[\kappa_3 + 1 : 2\kappa_3 - \kappa_1] \end{bmatrix}_{x_2^\vee}^{u_2^\vee},$$
$$\begin{bmatrix} b_2 \\ a_2^\vee \end{bmatrix}_{x_2^\sqcup}^{u_2^\sqcup} = \begin{bmatrix} b_3[\kappa_1 + 1 : \kappa_3] \\ a_3[\kappa_3 + 1 : 2\kappa_3 - \kappa_1]^\vee \end{bmatrix}_{x_2}^{u_2} \begin{bmatrix} b_3[\kappa_3 + 1 : 2\kappa_3 - \kappa_1] \\ a_3[\kappa_1 + 1 : \kappa_3]^\vee \end{bmatrix}_{x_2^\vee}^{u_2^\vee}.$$

7. *Furthermore, we have*

$$\begin{bmatrix} b_3 \\ a_3 \end{bmatrix}_{(x_1 + x_2)^\sqcup}^{(u_1 + u_2)^\sqcup} = \begin{bmatrix} b_1 \\ a_1 \end{bmatrix}_{x_1^\sqcup}^{u_1^\sqcup} \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}_{x_2^\sqcup}^{u_2^\sqcup},$$
$$\begin{bmatrix} b_3 \\ a_3^\vee \end{bmatrix}_{(x_1 + x_2)^\sqcup}^{(u_1 + u_2)^\sqcup} = \begin{bmatrix} b_1 \\ a_1^\vee \end{bmatrix}_{x_1^\sqcup}^{u_1^\sqcup} \begin{bmatrix} b_2 \\ a_2^\vee \end{bmatrix}_{x_2^\sqcup}^{u_2^\sqcup}.$$

8. *If μ_3 is wrapped, then μ_1 and μ_2 are wrapped.*

Proof. Proof of Claim 1. Let $\lambda_1 = 1$ and $\lambda_2 = \kappa_1$ in Proposition 9.2. Claim 1 follows from Claim 2 of Proposition 9.2.

Proof of Claim 2. First, Item 2 of Definition 9.1 gives equations (9.0.14) and (9.0.15). Second, Claim 1 of Proposition 9.2 implies equations (9.0.16) and (9.0.17). Third, Claim 3 of Proposition 9.2 imply equations (9.0.18), (9.0.19), and (9.0.20).

Proof of Claim 3. Claim 3 follows from equations (9.0.16) and (9.0.17).

We obtain Claims 4, 5, and 6 analogously, taking $\lambda_1 = \kappa_3 - \kappa_2 + 1$ and $\lambda_2 = \kappa_3$ in Proposition 9.2 instead.

Proof of Claim 7. Claim 7 follows from Claims 3 and 6.

Proof of Claim 8. Claim 8 holds by Claim 4 of Proposition 9.2. □

We state the following compatibility of squaring orders on finite sets $\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3$ of free indeterminates.

Lemma 9.4. *Suppose $\mathfrak{X}_3 = \mathfrak{X}_1 \cup \mathfrak{X}_2$. Then, squaring orders $\left\{\geq_{O_{\mathfrak{X}_3}}, >_{O_{\mathfrak{X}_3}}\right\}$ on $\mathfrak{X}_3$ are compatible to squaring orders $\left\{\geq_{O_{\mathfrak{X}_1}}, >_{O_{\mathfrak{X}_1}}\right\}$ on $\mathfrak{X}_1$.*

Proof. If $f >_{O_{\mathfrak{X}_1}} 0$, then $f >_{A_{\mathfrak{X}_3}} 0$ by $\mathfrak{X}_3 = \mathfrak{X}_1 \cup \mathfrak{X}_2$. A similar argument holds for $\geq_{O_{\mathfrak{X}_1}}$ and $\geq_{A_{\mathfrak{X}_3}}$. □

By the following proposition, we define the separable products of parcels as parcels.

Proposition 9.5. *Suppose parcels $\mathscr{F}_i = \Lambda(s, l_i, \succ_i, w_i, f_{i,s}, \phi_i, \rho_i, x_i, \mathfrak{X}_i)$ for $i \in [\![2]\!]$. For $\mathfrak{X}_3 = \mathfrak{X}_1 \cup \mathfrak{X}_2$ and $l_3 = l_1 + l_2$, consider*

$$f_{3,s} = \left(f_{3,s,m} = f_{1,s,m[1:l_1]} f_{2,s,m[l_1+1:l_3]} \in \mathbb{Q}(\mathfrak{X}_3)\right)_{m \in \mathbb{Z}^{l_3}}.$$

Let $x_3 = x_1 \mathbin{+\!\!+} x_2$, $w_3 = w_1 \mathbin{+\!\!+} w_2$, $\rho_3 = \rho_1 \mathbin{+\!\!+} \rho_2$, and $\phi_3(x_3) = \phi_1(x_1) \mathbin{+\!\!+} \phi_2(x_2)$. For the squaring orders $O_i = \{\succeq_i, \succ_i\}$ on $\mathfrak{X}_i$ of $i \in [\![2]\!]$, assume squaring orders $O_3 = \{\succeq_3, \succ_3\}$ on $\mathfrak{X}_3$ such that $O_3 \supseteq O_1, O_2$.

1. *Then, there exists a parcel $\mathscr{F}_3 = \Lambda(s, l_3, \succ_3, w_3, f_{3,s}, \phi_3, \rho_3, x_3, \mathfrak{X}_3)$.*
2. *For a fitting $\mu_3 = (s, l_3, m_3, n_3, k_3)$, let*

$$\mu_1 = (s, l_1, m_1, n_1, k_1) = \mathrm{ot}(l_1, \mu_3),$$
$$\mu_2 = (s, l_2, m_2, n_2, k_2) = \mathrm{ct}(l_2, \mu_3).$$

Then, we have

$$\begin{aligned}\Delta(\mathscr{F}_3)(s, l_3, w_3, m_3, n_3, k_3, \phi_3, \rho_3, x_3, \mathfrak{X}_3) = &\prod_{i \in [\![2]\!]} \Delta_L(\mathscr{F}_i)(s, l_i, w_i, m_i, n_i, k_i, \phi_i, \rho_i, x_i, \mathfrak{X}_i) \\ &- \prod_{i \in [\![2]\!]} \Delta_R(\mathscr{F}_i)(s, l_i, w_i, m_i, n_i, k_i, \phi_i, \rho_i, x_i, \mathfrak{X}_i).\end{aligned}$$

Proof. Proof of Claim 1. First, we prove that x_3 is $\succ_3$-admissible. If $g >_{x_{3,i}} 0$ for some $i \in [\![l_3]\!]$, then $g \succ_1 0$ or $g \succ_2 0$ by $x_3 = x_1 \mathbin{+\!\!+} x_2$ and the half $>_{x_{j,i}}$-$\succ_j$ implications of $j \in [\![2]\!]$. We deduce $f \succ_3 0$ by $O_3 \supseteq O_1, O_2$. In particular, the half $>_{x_{3,i}}$-$\succ_3$ implication holds for each $i \in [\![l_3]\!]$. Also, each $i \in [\![l_3]\!]$ satisfies the upper condition of $x_{3,i}$ on $\mathfrak{X}_3$ by $\mathfrak{X}_3 = \mathfrak{X}_1 \cup \mathfrak{X}_2$ and Lemma 9.4. Therefore, x_3 is $\succ_3$-admissible.

Second, $f_{3,s}$ is $\succ_3$-positive, because $f_{3,s,m} = f_{1,s,m[1:l_1]} f_{2,s,m[l_1+1:l_3]} \succ_3 0$ if $m \in [\![s]\!]^{l_3}$.

Third, we prove that ϕ_3 is a λ_3-mediator for $\lambda_3 = (s, l_3, w_3, \succ_3, \rho_3, x_3, \mathfrak{X}_3)$. Since $\mathscr{F}_1$ and $\mathscr{F}_2$ are parcels, ϕ_3 and λ_3 have the base positivity by $\phi_3(x_3) = \phi_1(x_1) \mathbin{+\!\!+} \phi_2(x_2)$ and Lemma 9.4. Furthermore, ϕ_3 and λ_3 have the base-shift positivity, since $m \in [\![s_3]\!]^{l_3}$ gives

$$B(s, l_3, w_3, m, \phi_3, \rho_3, x_3, \mathfrak{X}_3) = \prod_{i \in [\![2]\!]} B(s, l_i, w_i, u_i, \phi_i, \rho_i, x_i, \mathfrak{X}_i) \succ_3 0$$

for $u_1 = m[1 : l_1] \in [\![s]\!]^{l_1}$ and $u_2 = m[l_1 + 1 : l_3] \in [\![s]\!]^{l_2}$. Hence, ϕ_3 is a λ_3-mediator. Claim 1 follows now.

Proof of Claim 2. Let $a_3 = \nu(k_3)$ and $b_3 = \nu(m_3, n_3, k_3)$. Then, Claim 2 of Theorem 7.19 gives

$$\begin{aligned}&\Delta(\mathscr{F}_3)(s, l_3, w_3, m_3, n_3, k_3, \phi_3, \rho_3, x_3, \mathfrak{X}_3) \\ &= f_{3,s,m_3} f_{3,s,n_3^\vee} B(s, l_3, w_3, m_3, n_3^\vee, \phi_3, \rho_3, x_3, \mathfrak{X}_3) \begin{bmatrix} b_3 \\ a_3 \end{bmatrix}_{x_3^\sqcup}^{w_3^\sqcup} \\ &\quad - f_{3,s,m_3 \boxminus k_3} f_{3,s,(n_3 \boxplus k_3)^\vee} B(s, l_3, w_3, m_3 \boxminus k_3, (n_3 \boxplus k_3)^\vee, \phi_3, \rho_3, x_3, \mathfrak{X}_3) \begin{bmatrix} b_3 \\ a_3^\vee \end{bmatrix}_{x_3^\sqcup}^{w_3^\sqcup}.\end{aligned}$$

Hence, Claims 2 and 5 of Corollary 9.3 give Claim 2 by Claim 7 of Corollary 9.3. □

Definition 9.6. *Under the assumption of Proposition 9.5, we define the separable product $\mathscr{F}_1 \circ \mathscr{F}_2$ of $\mathscr{F}_1$ and $\mathscr{F}_2$ such that*

$$\mathscr{F}_1 \circ \mathscr{F}_2 = \mathscr{F}_3 = \Lambda(s, l_3, \succ_3, w_3, f_{3,s}, \phi_3, \rho_3, x_3, \mathfrak{X}_3).$$

In particular, each $m \in \mathbb{Z}^{l_3}$ satisfies

$$\mathscr{F}_{3,m} = \mathscr{F}_{1,m[1:l_1]}\mathscr{F}_{2,m[l_1+1:l_3]}.$$

In particular, separable products have the following merged-log-concavity.

Theorem 9.7. *Let $\mathscr{F}_i = \Lambda(s, l_i, \succ_i, w_i, f_{i,s}, \phi_i, \rho_i, x_i, \mathfrak{X}_i)$ for $i \in [\![3]\!]$ such that $\mathscr{F}_3 = \mathscr{F}_1 \diamond \mathscr{F}_2$. Consider squaring orders $O'_i = \{\succeq'_i, \succ'_i\}$ on $\mathfrak{X}_i$ for $i \in [\![3]\!]$ such that $O'_3 \supseteq O'_1, O'_2$. Let $\mathscr{F}_2$ be $\succeq'_2$-merged-log-concave. Then, we have the following.*

1. *$\mathscr{F}_3$ is $\succ'_3$-merged-log-concave if $\mathscr{F}_1$ is $\succ'_1$-merged-log-concave.*
2. *$\mathscr{F}_3$ is $\succeq'_3$-merged-log-concave if $\mathscr{F}_1$ is $\succeq'_1$-merged-log-concave.*

Proof. Proof of Claim 1. Suppose a fitting $\mu_3 = (s, l_3, m_3, n_3, k_3)$. Then, Claims 1 and 4 of Corollary 9.3 give fitting $(s, l_1, m_1, n_1, k_1) = \mathrm{ot}(l_1, \mu_3)$ and $(s, l_2, m_2, n_2, k_2) = \mathrm{ct}(l_2, \mu_3)$. Also, for $i \in [\![2]\!]$, let

$$L_i = \Delta_L(\mathscr{F}_i)(s, l_i, w_i, m_i, n_i, k_i, \phi_i, \rho_i, x_i, \mathfrak{X}_i),$$
$$R_i = \Delta_R(\mathscr{F}_i)(s, l_i, w_i, m_i, n_i, k_i, \phi_i, \rho_i, x_i, \mathfrak{X}_i).$$

Then, the merged-log-concavity of $\mathscr{F}_1$ and $\mathscr{F}_2$ gives $L_1 \succ'_1 R_1$ and $L_2 \succeq'_2 R_2$. Now, Claim 4 of Theorem 7.19 on $\mathscr{F}_1$ implies $R_1 \succeq_1 0$. Also, Claim 4 of Theorem 7.19 on $\mathscr{F}_2$ implies $L_2 \succ_2 0$, and either $R_2 \succ_2 0$ or $R_2 = 0$. Hence, Corollary 5.7 gives Claim 1 by the compatibilities $O'_3 \supseteq O'_1, O'_2$ and $O'_i \supseteq O_i$ of $i \in [\![2]\!]$.

Proof of Claim 2. Claim 2 holds by Claim 1d of Lemma 5.2, since $L_i \succeq'_3 R_i \succeq'_3 0$ for $i \in [\![2]\!]$. ☐

For separable products of parcels $\mathscr{F}_i$ of $i \in [\![d]\!]$, we write $\diamond_{i \in [\![d]\!]} \mathscr{F}_i$ for $\mathscr{F}_1 \diamond \ldots \diamond \mathscr{F}_d$. We state the following remark.

Remark 9.8. Assume the notation in Theorem 9.7. Furthermore, for simplicity, let $l_1 = l_2 = 1$, $\mathfrak{X}_1 = \mathfrak{X}_2$, $x_1 = x_2 = \iota^{l_1}(q)$, $O_1 = O_2 = O_3$, and ϕ_1 and ϕ_2 be canonical mediators. Then, we obtain width-two merged-log-concave parcels from width-one merged-log-concave parcels by $l_3 = 2$ and Theorem 9.7.

However, a width-two monomial parcel does not have to be the separable product of width-one monomial parcels. For instance, assume monomial indices (l_i, w_i, γ_i) for $i \in [\![3]\!]$ such that $\gamma_{3,1,1} > 0 > \gamma_{3,2,1}$. Then, $x_3 = \iota^{l_3}(q)$ gives

$$\Lambda(s, l_3, w_3, \succ, \Psi_{s,\gamma_3,q}, x_3, \mathfrak{X}_3) \neq \diamond_{i \in [\![2]\!]} \Lambda(s, l_i, w_i, \succ, \Psi_{s,\gamma_i,q}, x_i, \mathfrak{X}_i),$$

because $\gamma_{1,1,1}, \gamma_{2,1,1} \geq 0$ by the sum monomial conditions of (l_1, w_1, γ_1) and (l_2, w_2, γ_2).

We introduce the following multifold separable products for our later discussion.

Definition 9.9. *Let $\mathscr{F} = \Lambda(s, l, \succ, w, f_s, \phi, \rho, x, \mathfrak{X})$ and $d \in \mathbb{Z}_{\geq 1}$. Then, we define the d-fold separable product*

$$\mathscr{F}^{\diamond d} = \diamond_{i \in [\![d]\!]} \mathscr{F} = \Lambda(s, dl, \succ, w^{\sqcup d}, g_s, \phi^{\sqcup d}, \rho^{\sqcup d}, x^{\sqcup d}, \mathfrak{X})$$

such that $g_{s,m} = \prod_{\lambda \in [\![d]\!]} f_{s,m[(\lambda-1)l+1:\lambda l]}$ for each $m \in \mathbb{Z}^{dl}$.

10 Hadamard products

We introduce Hadamard products on parcels. This yields merged-log-concave parcels of higher weights from those of lower weights. To define the products, we state the following by segment additions in Definition 2.4.

Proposition 10.1. *Consider $l_1, l_2 \in \mathbb{Z}_{\geq 1}$ and $\lambda \in \mathrm{T}_{\leq}(2, l_1)$ such that $l_2 = \lambda_2 - \lambda_1 + 1$. Let $\mathscr{F}_i = \Lambda(s, l_i, w_i, \succ_i, f_{i,s}, \phi_i, \rho_i, x_i, \mathfrak{X}_i)$ for $i \in [\![2]\!]$ such that*

$$x_2 = x_1[\lambda_1 : \lambda_2],$$
$$\phi_2(x_2) = \phi_1(x_1)[\lambda_1 : \lambda_2],$$
$$\rho_2 = \rho_1[\lambda_1 : \lambda_2].$$

Let $O_i = \{\succeq_i, \succ_i\}$ on $\mathfrak{X}_i$ for $i \in [\![2]\!]$. Suppose squaring orders $O_3 = \{\succeq_3, \succ_3\}$ on $\mathfrak{X}_3 = \mathfrak{X}_1 \cup \mathfrak{X}_2$ such that $O_3 \Supset O_1, O_2$. Also, let

$$w_3 = w_1 +_\lambda w_2 \in \mathbb{Z}_{\geq 0}^{l_1},$$
$$f_{3,s} = \left(f_{3,s,m} = f_{1,s,m} f_{2,s,m[\lambda_1:\lambda_2]} \in \mathbb{Q}(\mathfrak{X}_3)\right)_{m \in \mathbb{Z}^{l_1}}.$$

Then, there is a parcel

$$\mathscr{F}_3 = \Lambda(s, l_1, w_3, \succ_3, f_{3,s}, \phi_1, \rho_1, x_1, \mathfrak{X}_3).$$

Proof. First, we prove that x_1 is $\succ_3$-admissible. Since $\mathscr{F}_1$ is a parcel, x_1 is $\succ_1$-admissible. This gives the half $>_{x_{1,i}}$-$\succ_3$ implication for each $x_{1,i}$ by $O_3 \Supset O_1$. Therefore, x_1 is $\succ_3$-admissible by Lemma 9.4, because we have the upper condition of each $x_{1,i}$ on $\mathfrak{X}_1$.

Second, $f_{3,s}$ is $\succ_3$-positive, because the compatibility $O_3 \Supset O_1, O_2$ implies $f_{3,s,m} = f_{1,s,m} f_{2,s,m[\lambda_1:\lambda_2]} \succ_3 0$ for $m \in [\![s]\!]^{l_1}$ by $f_{1,s,m} \succ_1 0$ and $f_{2,s,m[\lambda_1:\lambda_2]} \succ_2 0$.

Third, we prove that ϕ_1 is a μ-mediator for $\mu = (s, l_1, w_3, \succ_3, \rho_1, x_1, \mathfrak{X}_3)$. Now, $\phi_i(x_i)$ is a mediator of $\mathscr{F}_i$ for $i \in [\![2]\!]$. Hence, we have $\phi_1(x_1)_i^{w_{1,i}} >_{\mathrm{O}_{\mathfrak{X}_1}} 0$ for each $i \in [\![l_1]\!]$, and $\phi_1(x_1)_i^{w_{2,i-\lambda_1+1}} >_{\mathrm{O}_{\mathfrak{X}_2}} 0$ for each $i \in [\![\lambda_1, \lambda_2]\!]$. Hence, ϕ_1 and μ have the base positivity, since Lemma 9.4 implies

$$\phi_1(x_1)_i^{w_{3,i}} = \begin{cases} \phi_1(x_1)_i^{w_{1,i}} \phi_1(x_1)_i^{w_{2,i-\lambda_1+1}} >_{A_{\mathfrak{X}_3}} 0 \text{ for each } i \in [\![\lambda_1, \lambda_2]\!], \\ \phi_1(x_1)_i^{w_{1,i}} >_{A_{\mathfrak{X}_3}} 0 \text{ otherwise.} \end{cases}$$

Also, ϕ_1 and μ have the base-shift positivity, because each $m \in [\![s]\!]^{l_1}$ satisfies

$$\begin{aligned} B(s, l_1, w_3, m, \phi_1, \rho_1, x_1, \mathfrak{X}_3) &= \prod_{i \in [\![l_1]\!]} \frac{\phi_1(x_i^{\rho_{1,i}})^{w_{3,i} m_i} [m_i]!^{w_{3,i}}_{x_i^{\rho_{1,i}}}}{\phi_1(x_i)^{w_{3,i} m_i} [m_i]!^{w_{3,i}}_{x_i}} \\ &= B(s, l_1, w_1, m, \phi_1, \rho_1, x_1, \mathfrak{X}_1) \cdot B(s, l_2, w_2, m[\lambda_1 : \lambda_2], \phi_2, \rho_2, x_2, \mathfrak{X}_2) \\ &\succ_3 0. \end{aligned}$$

Hence, ϕ_1 is a μ-mediator. The assertion now follows. □

We then define the following product as a parcel.

Definition 10.2. *Under the assumption in Proposition 10.1, we define the Hadamard product*

$$\mathscr{F}_1 \circ_\lambda \mathscr{F}_2 = \mathscr{F}_2 \circ_\lambda \mathscr{F}_1 = \mathscr{F}_3 = \Lambda(s, l_1, w_3, \succ_3, f_{3,s}, \phi_1, \rho_1, x_1, \mathfrak{X}_3).$$

When $\mathscr{F}_1$ and $\mathscr{F}_2$ have the same widths, we simply write $\mathscr{F}_1 \circ \mathscr{F}_2 = \mathscr{F}_2 \circ \mathscr{F}_1$ for $\mathscr{F}_1 \circ_\lambda \mathscr{F}_2$.

We have the following merged-log-concavity of Hadamard products.

Theorem 10.3. *Let $\mathscr{F}_i = \Lambda(s, l_i, w_i, \succ_1, f_{i,s}, \phi_i, \rho_i, x_i, \mathfrak{X}_i)$ for $i \in [\![3]\!]$ such that*

$$\mathscr{F}_3 = \mathscr{F}_1 \circ_\lambda \mathscr{F}_2.$$

Consider squaring orders $O'_i = \{\succeq'_i, \succ'_i\}$ on $\mathfrak{X}_i$ for $i \in [\![3]\!]$ such that $O'_3 \Supset O'_1, O'_2$. Let $\mathscr{F}_2$ be $\succeq'_2$-merged-log-concave. Then, we have the following.

1. *$\mathscr{F}_3$ is $\succ'_3$-merged-log-concave if $\mathscr{F}_1$ is $\succ'_1$-merged-log-concave.*
2. *$\mathscr{F}_3$ is $\succeq'_3$-merged-log-concave if $\mathscr{F}_1$ is $\succeq'_1$-merged-log-concave.*

Proof. Proof of Claim 1. Assume a fitting $\mu_1 = (s, l_1, m_1, n_1, k_1)$. Then, Claim 2 of Proposition 9.2 gives the fitting $\mu_2 = (s, l_2, m_2, n_2, k_2) = t(\lambda, \mu_1)$. Also, let $\mu_3 = (s, l_3, m_3, n_3, k_3) = \mu_1$. Suppose $a_i = \nu(k_i)$, $b_i = \nu(m_i, n_i, k_i)$, and $y_i = x_i^{\rho_i}$ for $i \in [\![3]\!]$. Furthermore, consider $L_i, R_i \in \mathbb{Q}(\mathfrak{X}_i)$ for $i \in [\![3]\!]$ such that

$$\begin{aligned}
L_i &= \Delta_L(\mathscr{F}_i)(s, l_i, w_i, m_i, n_i, k_i, \phi_i, \rho_i, x_i, \mathfrak{X}_i) \\
&= f_{i,s,m_i} f_{i,s,n_i^\vee} B(s, l_i, w_i, m_i, n_i^\vee, \phi_i, \rho_i, x_i, \mathfrak{X}_i) \begin{bmatrix} b_i \\ a_i \end{bmatrix}_{y_i^\sqcup}^{w_i^\sqcup}, \\
R_i &= \Delta_R(\mathscr{F}_i)(s, l_i, w_i, m_i, n_i, k_i, \phi_i, \rho_i, x_i, \mathfrak{X}_i) \\
&= f_{i,s,m_i \boxminus k_i} f_{i,s,(n_i \boxplus k_i)^\vee} B(s, l_i, w_i, m_i \boxminus k_i, (n_i \boxplus k_i)^\vee, \phi_i, \rho_i, x_i, \mathfrak{X}_i) \begin{bmatrix} b_i \\ a_i^\vee \end{bmatrix}_{y_i^\sqcup}^{w_i^\sqcup}.
\end{aligned}$$

Let us prove

$$L_1 L_2 = L_3, \tag{10.0.1}$$
$$R_1 R_2 = R_3. \tag{10.0.2}$$

First, we establish

$$B(s, l_3, w_3, m_3, n_3^\vee, \phi_3, \rho_3, x_3, \mathfrak{X}_3) = \prod_{i \in [\![2]\!]} B(s, l_i, w_i, m_i, n_i^\vee, \phi_i, \rho_i, x_i, \mathfrak{X}_i). \tag{10.0.3}$$

Observe that each $i \in [\![\lambda]\!]$ gives

$$\begin{aligned}
b(s, w_{3,i}, m_{3,i}, \phi_{3,i}, \rho_{3,i}, x_{3,i}, \mathfrak{X}_3) &= b(s, w_{3,i}, m_{1,i}, \phi_{1,i}, \rho_{1,i}, x_{1,i}, \mathfrak{X}_3) \\
&= \frac{\phi_1(y_{1,i})^{m_{1,i} w_{3,i}} [m_{1,i}]_{y_{1,i}}^{w_{3,i}}}{\phi_1(x_{1,i})^{m_{1,i} w_{3,i}} [m_{1,i}]_{x_{1,i}}^{w_{3,i}}} \\
&= \frac{\phi_1(y_{1,i})^{m_{1,i} w_{1,i}} [m_{1,i}]_{y_{1,i}}^{w_{1,i}}}{\phi_1(x_{1,i})^{m_{1,i} w_{1,i}} [m_{1,i}]_{x_{1,i}}^{w_{1,i}}} \cdot \frac{\phi_2(y_{2,i-\lambda_1+1})^{m_{2,i-\lambda_1+1} w_{2,i-\lambda_1+1}} [m_{2,i-\lambda_1+1}]_{y_{2,i-\lambda_1+1}}^{w_{2,i-\lambda_1+1}}}{\phi_2(x_{2,i-\lambda_1+1})^{m_{2,i-\lambda_1+1} w_{2,i-\lambda_1+1}} [m_{2,i-\lambda_1+1}]_{x_{2,i-\lambda_1+1}}^{w_{2,i-\lambda_1+1}}} \\
&= b(s, w_{1,i}, m_{1,i}, \phi_{1,i}, \rho_{1,i}, x_{1,i}, \mathfrak{X}_1) \\
&\quad \cdot b(s, w_{2,i-\lambda_1+1}, m_{2,i-\lambda_1+1}, \phi_{2,i-\lambda_1+1}, \rho_{2,i-\lambda_1+1}, x_{2,i-\lambda_1+1}, \mathfrak{X}_2).
\end{aligned}$$

Similarly, since $n_2^\vee = n_1^\vee[\lambda_1 : \lambda_2]$, each $i \in [\![\lambda]\!]$ yields

$$\begin{aligned} b(s, w_{3,i}, (n_3^\vee)_i, \phi_{3,i}, \rho_{3,i}, x_{3,i}, \mathfrak{X}_3) &= b(s, w_{3,i}, (n_1^\vee)_i, \phi_{1,i}, \rho_{1,i}, x_{1,i}, \mathfrak{X}_3) \\ &= b(s, w_{1,i}, (n_1^\vee)_i, \phi_{1,i}, \rho_{1,i}, x_{1,i}, \mathfrak{X}_1) \\ &\quad \cdot b(s, w_{2,i-\lambda_1+1}, (n_2^\vee)_{i-\lambda_1+1}, \phi_{2,i-\lambda_1+1}, \rho_{2,i-\lambda_1+1}, x_{2,i-\lambda_1+1}, \mathfrak{X}_2). \end{aligned}$$

Hence, equation (10.0.3) follows.

Second, by Claim 3 of Proposition 9.2, we analogously obtain

$$B(s, l_1, w_3, m_3 \boxminus k_3, (n_3 \boxplus k_3)^\vee, \phi_3, \rho_3, x_3, \mathfrak{X}_3) = \prod_{i \in [\![2]\!]} B(s, l_i, w_i, m_i \boxminus k_i, (n_i \boxplus k_i)^\vee, \phi_i, \rho_i, x_i, \mathfrak{X}_i). \quad (10.0.4)$$

Third, since $w_3 = w_1 +_\lambda w_2$, Claim 1 of Proposition 9.2 implies

$$\begin{bmatrix} b_1 \\ a_1 \end{bmatrix}_{y_1^\sqcup}^{w_1^\sqcup} \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}_{y_2^\sqcup}^{w_2^\sqcup} = \begin{bmatrix} b_3 \\ a_3 \end{bmatrix}_{y_3^\sqcup}^{w_3^\sqcup}, \quad (10.0.5)$$

$$\begin{bmatrix} b_1 \\ a_1^\vee \end{bmatrix}_{y_1^\sqcup}^{w_1^\sqcup} \begin{bmatrix} b_2 \\ a_2^\vee \end{bmatrix}_{y_2^\sqcup}^{w_2^\sqcup} = \begin{bmatrix} b_3 \\ a_3^\vee \end{bmatrix}_{y_3^\sqcup}^{w_3^\sqcup}. \quad (10.0.6)$$

Therefore, equation (10.0.1) holds by equations (10.0.3) and (10.0.5). Also, equation (10.0.2) holds by equations (10.0.4) and (10.0.6).

We obtain the $\succ'_3$-merged-log-concavity of $\mathscr{F}_3$ as follows. We have $L_1 \succ'_1 R_1$ and $L_2 \succeq'_2 R_2$ by the merged-log-concavities of $\mathscr{F}_1$ and $\mathscr{F}_2$. Then, by $O'_3 \supseteq O'_1, O'_2$, Claim 4 of Theorem 7.19 gives $L_1 \succ'_3 R_1 \succeq'_3 0$, $L_2 \succ'_3 0$, $L_2 \succeq'_3 R_2$, and either $R_2 \succ'_3 0$ or $R_2 = 0$. Hence, equations (10.0.1) and (10.0.2) imply the $\succ'_3$-merged-log-concavity $L_3 \succ'_3 R_3$ of $\mathscr{F}_3$ by Corollary 5.7.

Proof of Claim 2. Because $L_1 \succeq'_1 R_1$ and $L_2 \succeq'_2 R_2$, Claim 2 follows from Claim 1d of Lemma 5.2. □

Hence, we obtain the following higher-weight strictly merged-log-concave parcels from weight-zero non-strictly merged-log-concave parcels.

Corollary 10.4. *Let $d \in \mathbb{Z}_{\geq 1}$. For each $i \in [\![d]\!]$, let $\mathscr{F}_i = \Lambda(s, l_i, \succ_i, f_{i,s}, \mathfrak{X}_i)$ be $\succeq_i$-merged-log-concave. Assume squaring orders $O = \{\succeq, \succ\}$ on $\mathfrak{X} = \cup_{i \in [\![d]\!]} \mathfrak{X}_i$ such that $O \supseteq \{\succeq_i, \succ_i\}$ for each $i \in [\![d]\!]$. Furthermore, consider*

$$\mathscr{G} = \Lambda(s, l, \succ, g_s, \mathfrak{X}) = \mathscr{F}_1 \diamond \mathscr{F}_2 \diamond \ldots \diamond \mathscr{F}_d.$$

Then, we have the following.

1. *For each $w \in \mathbb{Z}^l_{\geq 0}$, if $\Lambda(s, l, w, \succ, k_s, \phi, \rho, x, \mathfrak{X})$ is a $\succeq$-merged-log-concave parcel, then we obtain the $\succ$-merged-log-concave parcel*

$$\Lambda(s, l, w, \succ, k_s g_s, \phi, \rho, x, \mathfrak{X}).$$

2. *For each $w \in \mathbb{Z}^l_{\geq 1}$, if ϕ is a proper $(s, l, w, \succ, \rho, x, \mathfrak{X})$-mediator, then we obtain the $\succ$-merged-log-concave parcel*

$$\Lambda(s, l, w, \succ, g_s, \phi, \rho, x, \mathfrak{X}).$$

Proof. Proof of Claim 1. Claim 1 follows from Claim 1 of Theorem 10.3, because

$$\mathscr{G} = \Lambda(s,l,(0),\succ,g_s,\phi,\rho,x,\mathfrak{X})$$

is $\succeq$-merged-log-concave by Claim 2 of Theorem 9.7.

Proof of Claim 2. We have the constant parcel

$$\Lambda(s,l,w,\succ,1_{s,l},\phi,\rho,x,\mathfrak{X}),$$

which is $\succ$-merged-log-concave by Corollary 8.30. Hence, Claim 1 gives Claim 2. □

Remark 10.5. Assume an infinite gate $s \geq 0$ with $l = 1$ and $w \in \mathbb{Z}^l_{\geq 1}$. In Claim 2 of Corollary 10.4, $\Lambda(s,l,w,\succ,f_s,x,\mathfrak{X})$ is $\succ$-merged-log-concave, if the weight-zero $\Lambda(s,l,\succ,f_s,\mathfrak{X})$ is $\succeq$-merged-log-concave. However, the converse does not hold (see Section 1.9). For example, the positive-weight parcel

$$\Lambda(s,l,w,>_q,\Psi_{s,((\frac{1}{2},0,0)),q},x,\mathfrak{X})$$

is $>_q$-merged-log-concave, but the weight-zero parcel

$$\Lambda(s,l,>_q,\Psi_{s,((\frac{1}{2},0,0)),q},\mathfrak{X})$$

is not $\geq_q$-merged-log-concave by equation (1.0.1).

If there is $\succ$-admissible $x \in \mathbb{Q}(\mathfrak{X})^l$ for some $l \in \mathbb{Z}_{\geq 1}$, then $1 \succ 0$ by Claim 2 of Lemma 5.20. In particular, we have the constant parcel $\Lambda(s,l,\succ,1_{s,l},\mathfrak{X})$. Hence, we introduce the following multifold Hadamard products for our later discussion.

Definition 10.6. *Suppose $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ and $d \in \mathbb{Z}_{\geq 0}$. We define the d-fold Hadamard product $\mathscr{F}^{\circ d}$ such that*

$$\mathscr{F}^{\circ d} = \begin{cases} \Lambda(s,l,dw,\succ,f_s^d,\phi,\rho,x,\mathfrak{X}) \text{ if } d \geq 1, \\ \Lambda(s,l,\succ,1_{s,l},\mathfrak{X}) \text{ otherwise}. \end{cases}$$

11 Weight-zero merged-log-concavity

First, we compare the weight-zero merged-log-concavity with the strong q-log-concavity and q-log-concavity in Definition 1.3. In particular, strong q-log-concave polynomials are weight-zero $\geq_q$-merged-log-concave parcels in a suitable setting. Hence, strongly q-log-concave polynomials give higher-weight merged-log-concave parcels by Corollary 10.4. Second, we discuss some analogs of conjectures in Section 8 by q-numbers. Third, we give log-concavity conjectures on weight-zero parcels of q-Starling polynomials, Ramanujan polynomials, and Bessel polynomials.

11.1 Strong q-log-concavity

To compare the weight-zero merged-log-concavity with the strong q-log-concavity, we introduce an intermediate notion by the following fitting tuples.

Lemma 11.1. *Let $l \in \mathbb{Z}_{\geq 1}$. Suppose $n,m \in [\![s]\!]^l$ such that $n^\vee \geq m$ and $m \sqcup (n+1) \in \mathbb{Z}^{2l}$ is increasing. Then, $\mu = (s,l,m,n,k)$ is fitting for $k = \iota^l(0) \sqcup (1) \sqcup \iota^{l-1}(0)$.*

Proof. Let $a = \nu(k)$ and $b = \nu(m,n,k)$. Then, $a = \iota^l(0) \sqcup \iota^l(1)$ and $b = m \sqcup (n+1)$. This deduces the slope conditions of μ, since $n^\vee \geq m$ implies $b_l = m_l < n_1 + 1 = b_{l+1}$. □

Taking fitting (s,l,m,n,k) that have the smallest $\sigma(k)_1$, we introduce the following intermediate notion by parcels and squaring orders.

Definition 11.2. *Let $l \in \mathbb{Z}_{\geq 1}$ and $\mathscr{F} = \Lambda(s,l,\succ,f_s,\mathfrak{X})$. Suppose squaring orders $O' = \{\succeq', \succ'\} \supseteq \{\succeq, \succ\}$.*

1. *$\mathscr{F}$ is strongly $\succ'$-multi-log-concave if*

$$\mathscr{F}_m \mathscr{F}_{n^\vee} - \mathscr{F}_{m-1} \mathscr{F}_{(n+1)^\vee} \succ' 0$$

whenever $n,m \in [\![s]\!]^l$, $n^\vee \geq m$, and $m \sqcup (n+1) \in \mathbb{Z}^{2l}$ is increasing.

2. *$\mathscr{F}$ is strongly $\succeq'$-multi-log-concave if*

$$\mathscr{F}_m \mathscr{F}_{n^\vee} - \mathscr{F}_{m-1} \mathscr{F}_{(n+1)^\vee} \succeq' 0$$

whenever $n,m \in [\![s]\!]^l$, $n^\vee \geq m$, and $m \sqcup (n+1) \in \mathbb{Z}^{2l}$ is increasing.

We state the following telescoping lemma to compare the strong multi-log-concavity and the merged-log-concavity.

Lemma 11.3. *Let $l \in \mathbb{Z}_{\geq 1}$ and $\mathscr{F} = \Lambda(s,l,\succ,f_s,\mathfrak{X})$. Then, $\mathscr{F}$ is strongly $\succ'$-multi-log-concave if and only if*

$$\mathscr{F}_m \mathscr{F}_{n^\vee} - \mathscr{F}_{m-k} \mathscr{F}_{(n+k)^\vee} \succ' 0 \tag{11.1.1}$$

whenever $m,n \in [\![s]\!]^l$, $n^\vee \geq m$, $k \in \mathbb{Z}_{\geq 1}$, and $m \sqcup (n+k)$ is increasing.

Proof. The if part holds by $k = 1$. We prove the only if part, assuming inequality (11.1.1) for $k = 1$. Let $m,n \in [\![s]\!]^l$ such that $n^\vee \geq m$ and $m \sqcup (n+1)$ is increasing.

First, suppose $i \in \mathbb{Z}_{\geq 0}$ such that $m - i \geq s_1$ and $n + i \leq s_2$. Then, $m-i, n+i \in [\![s]\!]^l$. Also, $n^\vee \geq m$ implies $(n+i)^\vee = n^\vee + i \geq m - i$. Furthermore, since $(m-i) \sqcup (n+i+1)$ is increasing, $\mathscr{F}_{m-i}\mathscr{F}_{(n+i)^\vee} - \mathscr{F}_{m-i-1}\mathscr{F}_{(n+i+1)^\vee} \succ' 0$.

Second, suppose $i \in \mathbb{Z}_{\geq 0}$ such that $m - i < s_1$ or $n + i > s_2$. Then, $m - i - 1 < s_1$ or $n + i + 1 > s_2$. Consequently, $\mathscr{F}_{m-i}\mathscr{F}_{(n+i)^\vee} - \mathscr{F}_{m-i-1}\mathscr{F}_{(n+i+1)^\vee} = 0$. In particular, each $k \in \mathbb{Z}_{\geq 1}$ yields

$$\begin{aligned}\mathscr{F}_m \mathscr{F}_{n^\vee} - \mathscr{F}_{m-k} \mathscr{F}_{(n+k)^\vee} &= \sum_{i \in [\![0,k-1]\!]} \left(\mathscr{F}_{m-i}\mathscr{F}_{(n+i)^\vee} - \mathscr{F}_{m-i-1}\mathscr{F}_{(n+i+1)^\vee}\right) \\ &\succeq' \mathscr{F}_m \mathscr{F}_{n^\vee} - \mathscr{F}_{m-1}\mathscr{F}_{(n+1)^\vee} \\ &\succ' 0.\end{aligned}$$

□

We have the following comparison by Lemma 11.1 and the telescoping lemma.

Proposition 11.4. *Let $l \in \mathbb{Z}_{\geq 1}$ and $\mathscr{F} = \Lambda(s,l,\succ,f_s,\mathfrak{X})$. Then, we have the following.*

1. *$\mathscr{F}$ is strongly $\succ'$-multi-log-concave if $\mathscr{F}$ is $\succ'$-merged-log-concave.*

2. *$\mathscr{F}$ is strongly $\succeq'$-multi-log-concave if $\mathscr{F}$ is $\succeq'$-merged-log-concave.*

Furthermore, assume $l = 1$. Then, we have the following.

(a) $\mathscr{F}$ is $\succ'$-merged-log-concave if and only if $\mathscr{F}$ is strongly $\succ'$-multi-log-concave.

(b) $\mathscr{F}$ is $\succeq'$-merged-log-concave if and only if $\mathscr{F}$ is strongly $\succeq'$-multi-log-concave.

Proof. Proof of Claim 1. Consider $m,n \in [\![s]\!]^l$ such that $n^\vee \geq m$ and $m \mathbin{+\!\!+} (n+1) \in \mathbb{Z}^{2l}$ is increasing. Let $\mu = (s,l,m,n,k)$ for $k = \iota^l(0) \mathbin{+\!\!+} (1) \mathbin{+\!\!+} \iota^{l-1}(0)$. By Lemma 11.1, we deduce $\Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X}) = f_{s,m} f_{s,n^\vee} - f_{s,m \boxminus k} f_{s,(n \boxplus k)^\vee} \succ' 0$. Claim 1 follows, since $m \boxminus k = m-1$ and $n \boxplus k = n+1$.

Proof of Claim 2. We obtain Claim 2 analogously, replacing $\succ'$ with $\succeq'$.

Proof of Claim (a). First, we prove the only if part. By Lemma 3.10, $\mathscr{F}$ is $\succ'$-merged-log-concave if and only if

$$\Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X}) = f_{s,m} f_{s,n} - f_{s,m-k_2} f_{s,n+k_2} \succ' 0 \tag{11.1.2}$$

whenever

$$m,n \in [\![s]\!]^l,\ k = (k_1,k_2) \geq (0,1),\ \text{and } n_1 + k_2 > m_1. \tag{11.1.3}$$

Therefore, we obtain the strong $\succ'$-multi-log-concavity of $\mathscr{F}$, since conditions (11.1.3) for $k_2 = 1$ imply $m,n \in [\![s]\!]^l$ such that $n^\vee \geq m$ and $m \mathbin{+\!\!+} (n+1)$ is increasing.

Second, we prove the if part. Consider m,n,k that satisfy conditions (11.1.3). If $n \geq m$, then inequality (11.1.2) holds by Lemma 11.3. If $n < m$, then let $m' = n$, $n' = m$, and $k'_2 = n_1 + k_2 - m_1 \in \mathbb{Z}_{\geq 1}$. We derive

$$\begin{aligned} n' &> m', \\ m' - k'_2 &= n - (n + k_2 - m) = m - k_2, \\ n' + k'_2 &= m + (n + k_2 - m) = n + k_2. \end{aligned}$$

Lemma 11.3 implies inequality (11.1.2), since

$$f_{s,m} f_{s,n} - f_{s,m-k_2} f_{s,n+k_2} = f_{s,m'} f_{s,n'} - f_{s,m'-k'_2} f_{s,n'+k'_2}.$$

Proof of Claim (b). Claim (b) follows from an analogous argument, where we replace $\succeq'$ with $\succ'$. □

For our convenience, we adopt the following notation.

Definition 11.5. *If $f = (f_m \in \mathbb{Q}(\mathfrak{X}))_{m \in \mathbb{Z}^1}$, then let* $\mathrm{Unt}(f) = \left(f_{(i)}\right)_{i \in \mathbb{Z}}$*. Conversely, if $f = (f_i \in \mathbb{Q}(\mathfrak{X}))_{i \in \mathbb{Z}}$, then let* $\mathrm{Tup}(f) = (f_i)_{(i) \in \mathbb{Z}^1}$.

We obtain the following comparison between the strong q-log-concavity and the width-one and weight-zero merged-log-concavity.

Corollary 11.6. *Assume a gate $s \geq 0$ and $>_q$-admissible $q \in \mathbb{Q}(\mathfrak{X})$. Let $f_s = (f_{s,i} \in \mathbb{Z}_{\geq 0}[q])_{i \in \mathbb{Z}}$ such that $f_{s,i} >_q 0$ for $i \in [\![s]\!]$. Also, let $g_s = \mathrm{Tup}(f_s)$, $l = 1$, and $\mathscr{F} = \Lambda(s,l,>_q,g_s,\mathfrak{X})$. Then, f_s is strongly q-log-concave if and only if $\mathscr{F}$ is $\geq_q$-merged-log-concave.*

Proof. The assertion follows from Claim (b) of Proposition 11.4, since the strong $\geq_q$-multi-log-concavity of $\mathscr{F}$ is equivalent to the strong q-log-concavity of f_s by $l = 1$. □

11.2 q-log-concavity

We state the following lemma to compare the q-log-concavity with the merged-log-concavity.

Lemma 11.7. *Let $l \in \mathbb{Z}_{\geq 1}$ and $m \in \mathbb{Z}^l$. Then, m is flat if $m^\vee \geq m$ and $m \sqcup (m+1) \in \mathbb{Z}^{2l}$ is increasing.*

Proof. Since $m \sqcup (m+1)$ is increasing, we have $m_1 \leq \ldots \leq m_l \leq m_1 + 1$. Also, $m_1 \geq m_l$ by $m^\vee \geq m$. $\square$

By these flat tuples, we give the following intermediate notation as before for the strong q-log-concavity.

Definition 11.8. *Let $l \in \mathbb{Z}_{\geq 1}$ and $\mathscr{F} = \Lambda(s, l, \succ, f_s, \mathfrak{X})$. Suppose squaring orders $\{\succeq', \succ'\} \supseteq \{\succeq, \succ\}$.*

1. *$\mathscr{F}$ is $\succ'$-multi-log-concave if*

$$\mathscr{F}_m \mathscr{F}_m - \mathscr{F}_{m-1} \mathscr{F}_{m+1} \succ' 0$$

for any flat $m \in [\![s]\!]^l$.

2. *$\mathscr{F}$ is $\succeq'$-multi-log-concave if*

$$\mathscr{F}_m \mathscr{F}_m - \mathscr{F}_{m-1} \mathscr{F}_{m+1} \succeq' 0$$

for any flat $m \in [\![s]\!]^l$.

We have the following comparison between the multi-log-concavity and the merged-log-concavity.

Proposition 11.9. *Let $l \in \mathbb{Z}_{\geq 1}$ and $\mathscr{F} = \Lambda(s, l, \succ, f_s, \mathfrak{X})$. Then, we have the following.*

1. *$\mathscr{F}$ is $\succ'$-multi-log-concave if $\mathscr{F}$ is $\succ'$-merged-log-concave.*
2. *$\mathscr{F}$ is $\succeq'$-multi-log-concave if $\mathscr{F}$ is $\succeq'$-merged-log-concave.*

Proof. By Lemma 11.7, the strong $\succ'$-multi-log-concavity and the strong $\succeq'$-multi-log-concavity imply the $\succ'$-multi-log-concavity and the $\succeq'$-multi-log-concavity, respectively. Claims 1 and 2 follow from Claims 1 and 2 of Proposition 11.4. $\square$

Furthermore, we have the following comparison between the q-log-concavity the merged-log-concavity.

Corollary 11.10. *Let $l = 1$. Suppose a $\geq_q$-merged-log-concave $\mathscr{F} = \Lambda(s, l, >_q, f_s, \mathfrak{X})$ such that $f_s = (f_{s,m} \in \mathbb{Z}_{\geq 0}[q])_{m \in \mathbb{Z}^l}$. Then, $g_s = \mathrm{Unt}(f_s)$ is q-log-concave.*

Proof. Since $l = 1$, the assertion holds by Claim 2 of Proposition 11.9. $\square$

In particular, if $\mathscr{F}$ is $>_q$-merged-log-concave, then $g_{s,i}^2 - g_{s,i-1} g_{s,i+1} >_q 0$ for $i \in [\![s]\!]$, which give almost strictly unimodal sequences.

It is possible to modify the notion of merged-log-concavity to completely extend the q-log-concavity and strong q-log-concavity. However, in this manuscript, a parcel demands the $\succ$-positivity of its numerators to give unimodal sequences and almost unimodal sequences in Definitions 1.1 and 1.17. Also, we compute merged determinants by fitting tuples to obtain polynomials with positive integer coefficients.

11.3 On some analogs of conjectures in Section 8

We introduce the following weight-zero parcels.

Definition 11.11. *Assume a gate $s \geq 1$, $l \in \mathbb{Z}_{\geq 1}$, and $q \in \mathbb{Q}(\mathfrak{X})$. Let $\chi_{s,l,q} = \left(\chi_{s,l,q,m} \in \mathbb{Z}_{\geq 0}[q]\right)_{m \in \mathbb{Z}^l}$ such that*

$$\chi_{s,l,q,m} = \begin{cases} [m]_q \text{ if } m \in [\![s]\!]^l, \\ 0 \text{ otherwise}. \end{cases}$$

If q is $\succ$-admissible, then we call $\Lambda(s,l,\succ,\chi_{s,l,q},\mathfrak{X})$ a q-number parcel.

We confirm some analogs of Conjectures 8.48 and 8.59 by the q-number parcels. For this, we adopt the following integer notation.

Definition 11.12. *Suppose $m_1, n_1, \lambda \in \mathbb{Z}$. Then, let*

$$I(\lambda, m_1, n_1) = \min(\lambda + 1, m_1, n_1, m_1 + n_1 - 1 - \lambda) \in \mathbb{Z}.$$

We examine these integers $I(\lambda, m_1, n_1)$ for the products of q-numbers.

Lemma 11.13. *Let $m_1, n_1 \in \mathbb{Z}_{\geq 1}$ and $\lambda \in \mathbb{Z}$.*

1. *If $m_1 \leq n_1$, then*

$$I(\lambda, m_1, n_1) = \begin{cases} \lambda + 1 \text{ if } \lambda \in [\![0, m_1 - 1]\!], & (11.3.1) \\ m_1 \text{ if } \lambda \in [\![m_1 - 1, n_1 - 1]\!], & (11.3.2) \\ m_1 + n_1 - 1 - \lambda \text{ if } \lambda \in [\![n_1 - 1, m_1 + n_1 - 2]\!]. & (11.3.3) \end{cases}$$

2. *We have $[m_1]_q [n_1]_q = \sum_{\lambda \in [\![0, m_1 + n_1 - 2]\!]} I(\lambda, m_1, n_1) q^\lambda$.*
3. *Assume $n_1 + k_2 > m_1$ for some $k_2 \in \mathbb{Z}_{\geq 1}$.*
   (a) *We have $I(\lambda, m_1, n_1) \geq I(\lambda, m_1 - k_2, n_1 + k_2)$.*
   (b) *If $m_1 - k_2 \geq 1$, then there is $\lambda \in \mathbb{Z}$ such that*

$$m_1 + n_1 - 2 \geq n_1 - 1 \geq \lambda \geq m_1 - k_2 \geq 0, \quad (11.3.4)$$
$$I(\lambda, m_1, n_1) > I(\lambda, m_1 - k_2, n_1 + k_2). \quad (11.3.5)$$

Proof. Proof of Claim 1. First, if $\lambda \in [\![0, m_1 - 1]\!]$, then equation (11.3.1) follows from

$$\lambda + 1 \leq m_1 \leq n_1 \leq n_1 + m_1 - (\lambda + 1) = m_1 + n_1 - 1 - \lambda.$$

Second, if $\lambda \in [\![m_1 - 1, n_1 - 1]\!]$, then equation (11.3.2) follows from

$$m_1 \leq \lambda + 1 \leq n_1,$$
$$m_1 + n_1 - (1 + \lambda) \geq m_1.$$

Third, if $\lambda \in [\![n_1 - 1, m_1 + n_1 - 2]\!]$, then equation (11.3.3) holds by

$$\lambda + 1 \geq n_1 \geq m_1 \geq m_1 + n_1 - 1 - \lambda.$$

Proof of Claim 2. We have

$$[m_1]_q[n_1]_q = \sum_{t_1 \in [\![0,m_1-1]\!], t_2 \in [\![0,n_1-1]\!]} q^{t_1+t_2}. \tag{11.3.6}$$

Assume $n_1 \geq m_1$ without loss of generality. For $\lambda \in \mathbb{Z}$, suppose (t_1,t_2) in equation (11.3.6) such that $\lambda = t_1+t_2$. Claim 1 implies Claim 2, since the right-hand sides of equations (11.3.1), (11.3.2), and (11.3.3) coincide with the following numbers of choices of (t_1,t_2).

First, if $\lambda \in [\![0,m_1-1]\!]$, then $\lambda \leq n_1-1$ gives the $\lambda+1$ choices $(\lambda,0),(\lambda-1,1),\ldots,(0,\lambda)$. Second, if $\lambda \in [\![m_1-1,n_1-1]\!]$, then $m_1 \geq 1$ gives the m_1 choices $(m_1-1,\lambda-(m_1-1)),\ldots,(0,\lambda)$. Third, if $\lambda \in [\![n_1-1,m_1+n_1-2]\!]$, then we have the $m_1+n_1-1-\lambda$ choices $(m_1-1,\lambda-(m_1-1)),\ldots,(\lambda-(n_1-1),n_1-1)$, since

$$\begin{aligned} 0 \leq n_1-m_1 \leq (\lambda+1)-m_1 = \lambda-(m_1-1) &\leq n_1-1, \\ \lambda-(n_1-1) &\leq m_1-1. \end{aligned}$$

Proof of Claim 3a. If $m_1 \leq n_1$, then $m_1-k_2 < m_1 \leq n_1$. If $m_1 > n_1$, then $m_1 > n_1 > m_1-k_2$. In either case, we have

$$I(\lambda,m_1,n_1) \geq \min(\lambda+1,m_1-k_2,n_1+k_2,m_1+n_1-1-\lambda) = I(\lambda,m_1-k_2,n_1+k_2).$$

Proof of Claim 3b. First, we have $\lambda \in \mathbb{Z}$ in inequality (11.3.4), since $m_1 \geq k_2+1 \geq 1$ gives $m_1+n_1-2 \geq n_1-1$ and $n_1 > m_1-k_2$ gives $n_1-1 \geq m_1-k_2$.

Second, we prove that inequality (11.3.4) implies inequality (11.3.5). In inequality (11.3.4), we have $n_1-1 \geq \lambda$. This gives

$$\begin{gathered} n_1 \geq \lambda+1, \\ m_1+n_1-1-\lambda \geq m_1 > m_1-k_2. \end{gathered}$$

Also, in inequality (11.3.4), we have $\lambda \geq m_1-k_2$. This gives

$$\lambda+1 > m_1-k_2.$$

By $m_1 > m_1-k_2$ and $n_1+k_2 > n_1$, we deduce

$$I(\lambda,m_1,n_1) > m_1-k_2 = I(\lambda,m_1-k_2,n_1+k_2).$$

□

Suppose $\mathscr{F} = \Lambda(s,l,\succ,\chi_{s,l,q},\mathfrak{X})$. Then, $\mathscr{F}$ is $\geq_q$-merged-log-concave by Claim 2 of Theorem 9.7, Corollary 11.6, and the strong q-log-concavity of q-numbers [Sag, Lemma 2.1]. Furthermore, we prove the following $>_q$-merged-log-concavity of $\mathscr{F}$ by palindromic unimodal merged determinants. In particular, these merged determinants we confirm an analog of Conjecture 8.59.

Proposition 11.14. *For $l \in \mathbb{Z}_{\geq 1}$, let $\mathscr{F} = \Lambda(s,l,\succ,\chi_{s,l,q},\mathfrak{X})$. Suppose a fitting $\mu = (s,l,m,n,k)$. Then, we have the following.*

1. $\Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X}) >_q 0$.
2. $\Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X})$ *is a palindromic unimodal q-polynomial.*

Proof. Assume that μ is unwrapped. This implies $\Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X}) = \chi_{s,l,q,m}\chi_{s,l,q,n}$. Then, Claim 1 holds by $m,n \geq 1$. Also, Claim 2 holds by Claim 1 of Proposition 8.15. Let us assume that μ is wrapped.

Proof of Claim 1. First, assume $l = 1$. Since μ is wrapped, Lemma 3.10 implies

$$n_1 + k_2 > m_1 > m_1 - k_2 \geq s_1 \geq 1. \tag{11.3.7}$$

Claim 1 holds by equation (11.3.8), since Claims 2 and 3 of Lemma 11.13 give

$$\begin{aligned}
[m_1]_q[n_1]_q &= \sum_{\lambda \in [\![0, m_1+n_1-2]\!]} I(\lambda, m_1, n_1) q^{\lambda} \\
&>_q \sum_{\lambda \in [\![0, m_1+n_1-2]\!]} I(\lambda, m_1 - k_2, n_1 + k_2) q^{\lambda} \\
&= [m_1 - k_2]_q [n_1 + k_2]_q.
\end{aligned}$$

Second, Claim 1 for $l \in \mathbb{Z}_{\geq 1}$ follows from Claim 1 of Theorem 9.7.

Proof of Claim 2. By Claim 1 of Proposition 8.15, $\chi_{s,l,q,m}\chi_{s,l,q,n}$ and $\chi_{s,l,q,m \boxminus k}\chi_{s,l,q,n \boxplus k}$ are palindromic unimodal q-polynomials. Also,

$$\begin{aligned}
\operatorname{ord}_q(\chi_{s,l,q,m}\chi_{s,l,q,n}) &= 0 = \operatorname{ord}_q(\chi_{s,l,q,m \boxminus k}\chi_{s,l,q,n \boxplus k}), \\
\deg_q(\chi_{s,l,q,m}\chi_{s,l,q,n}) &= \sum m + \sum n - 2l = \deg_q(\chi_{s,l,q,m \boxminus k}\chi_{s,l,q,n \boxplus k}).
\end{aligned}$$

We deduce Claim 2, since $\Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X})$ is a difference of the palindromic unimodal q-polynomials with the same orders and degrees. $\square$

As in the following, $\Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X})$ is not necessarily a log-concave q-polynomial.

Example 11.15. Let $l = 2$, $m = (5,2)$, $n = (2,5)$, and $k = (0,0,1,0)$. Then, by $m \boxminus k = (4,1)$ and $n \boxplus k = (3,6)$, $\mathscr{F} = \Lambda(s,l,\succ,\chi_{s,l,q},\mathfrak{X})$ gives the following unimodal and non-log-concave q-polynomial:

$$\begin{aligned}
\Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X}) &= [5]_q[2]_q[5]_q[2]_q - [4]_q[1]_q[6]_q[3]_q \\
&= q^9 + 2q^8 + 3q^7 + 5q^6 + 6q^5 + 5q^4 + 3q^3 + 2q^2 + q.
\end{aligned}$$

But, when $l = 1$, we state the following log-concave q-polynomials for $\mathscr{F} = \Lambda(s,l,\succ,\chi_{s,l,q},\mathfrak{X})$. Furthermore, the gap-free property of these q-polynomials confirms an analog of Conjecture 8.48.

Proposition 11.16. *For $l = 1$, let $\mathscr{F} = \Lambda(s,l,\succ,\chi_{s,l,q},\mathfrak{X})$. Suppose a fitting $\mu = (s,l,m,n,k)$. Then, we have the following.*

1. *$\Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X})$ is a log-concave q-polynomial.*
2. *$\Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X}) >_{q,d} 0$.*

Proof. We have

$$\Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X}) = \chi_{s,l,q,m}\chi_{s,l,q,n} - \chi_{s,l,q,m-k_2}\chi_{s,l,q,n+k_2}. \tag{11.3.8}$$

If μ is unwrapped, then $\Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X}) = \chi_{s,l,q,m}\chi_{s,l,q,n}$. This gives Claims 1 and 2 by Claims 1 and 2 of Lemma 11.13. Let us assume that μ is wrapped.

Proof of Claim 1. For each $\lambda \in \mathbb{Z}$, we define

$$P(\lambda, m_1, n_1, k_2) = I(\lambda, m_1, n_1) - I(\lambda, m_1 - k_2, n_1 + k_2),$$
$$\delta(P)(\lambda, m_1, n_1, k_2) = P(\lambda, m_1, n_1, k_2)^2 - P(\lambda - 1, m_1, n_1, k_2)P(\lambda + 1, m_1, n_1, k_2).$$

For $\lambda \in [\![0, m_1 + n_1 - 2]\!]$, we prove

$$\delta(P)(\lambda, m_1, n_1, k_2) \geq 0. \tag{11.3.9}$$

First, assume $m_1 \leq n_1$. Claim 1 of Lemma 11.13 and inequality (11.3.7) give $I(\lambda, m_1, n_1)$, $I(\lambda, m_1 - k_2, n_1 + k_2)$, and $P(\lambda, m_1, n_1, k_2)$ in Figure 4.

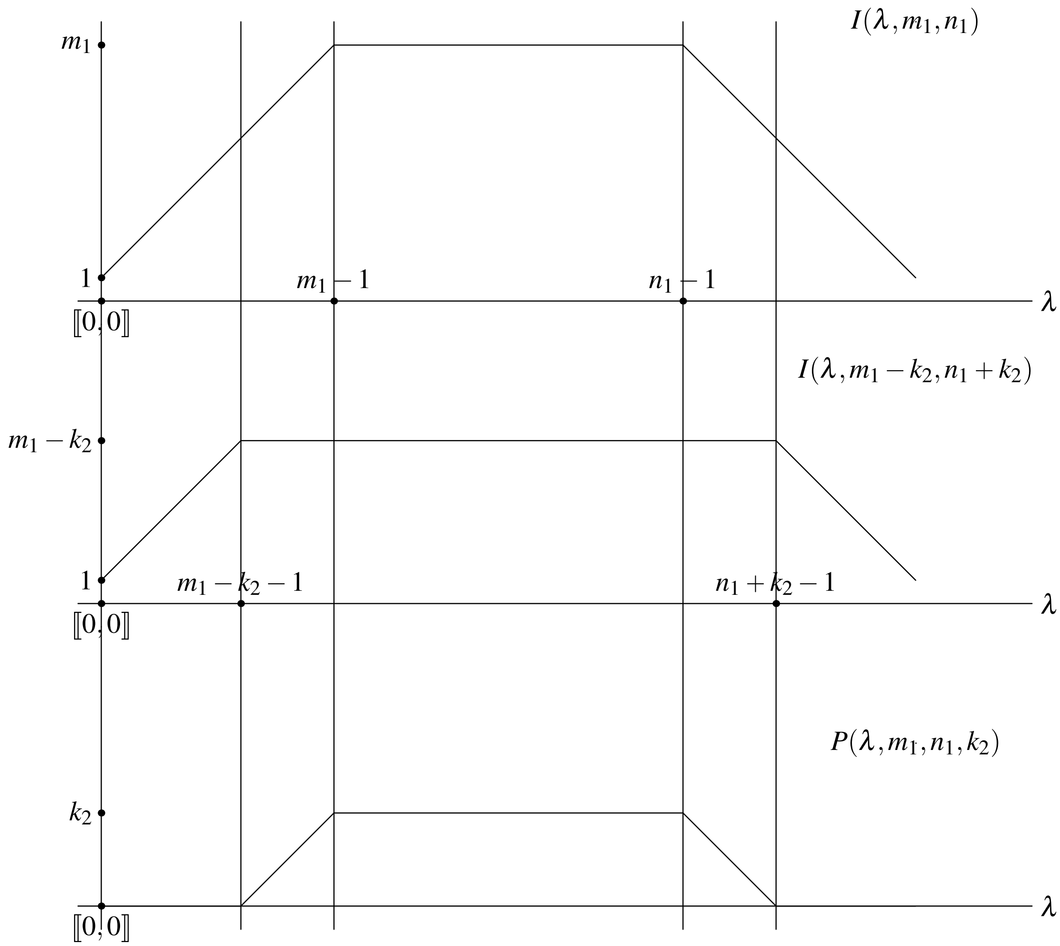


Figure 4: $P(\lambda, m_1, n_1, k_2)$ when $m_1 \leq n_1$

By Figure 4, we derive the following equation:

$$P(\lambda,m_1,n_1,k_2)=\begin{cases}0 & \text{if } \lambda\in[\![0,m_1-k_2-1]\!],\\ \lambda+1-m_1+k_2 & \text{if } \lambda\in[\![m_1-k_2-1,m_1-1]\!],\\ k_2 & \text{if } \lambda\in[\![m_1-1,n_1-1]\!],\\ n_1+k_2-1-\lambda & \text{if } \lambda\in[\![n_1-1,n_1+k_2-1]\!],\\ 0 & \text{if } \lambda\in[\![n_1+k_2-1,m_1+n_1-2]\!].\end{cases} \tag{11.3.10}$$

By equation (11.3.10), $m_1<n_1$ implies the following equation:

$$\delta(P)(\lambda,m_1,n_1,k_2)=\begin{cases}0 & \text{if } \lambda\in[\![0,m_1-k_2-1]\!],\\ 1 & \text{if } \lambda\in[\![m_1-k_2,m_1-2]\!],\\ k_2 & \text{if } \lambda=m_1-1,\\ 0 & \text{if } \lambda\in[\![m_1,n_1-2]\!],\\ k_2 & \text{if } \lambda=n_1-1,\\ 1 & \text{if } \lambda\in[\![n_1,n_1+k_2-2]\!],\\ 0 & \text{if } \lambda\in[\![n_1+k_2-1,m_1+n_1-2]\!].\end{cases}$$

If $m_1=n_1$, then we have the following equation:

$$\delta(P)(\lambda,m_1,n_1,k_2)=\begin{cases}0 & \text{if } \lambda\in[\![0,m_1-k_2-1]\!],\\ 1 & \text{if } \lambda\in[\![m_1-k_2,m_1-2]\!],\\ 2k_2-1 & \text{if } \lambda=m_1-1=n_1-1,\\ 1 & \text{if } \lambda\in[\![n_1,n_1+k_2-2]\!],\\ 0 & \text{if } \lambda\in[\![n_1+k_2-1,m_1+n_1-2]\!].\end{cases}$$

Therefore, inequality (11.3.9) follows from $k_2\geq 1$ for the fitting μ.

Second, assume $m_1>n_1$. Then, Claim 1 of Lemma 11.13 and inequality (11.3.7) give $I(\lambda,m_1,n_1)$, $I(\lambda,m_1-k_2,n_1+k_2)$, and $P(\lambda,m_1,n_1,k_2)$ in Figure 5.

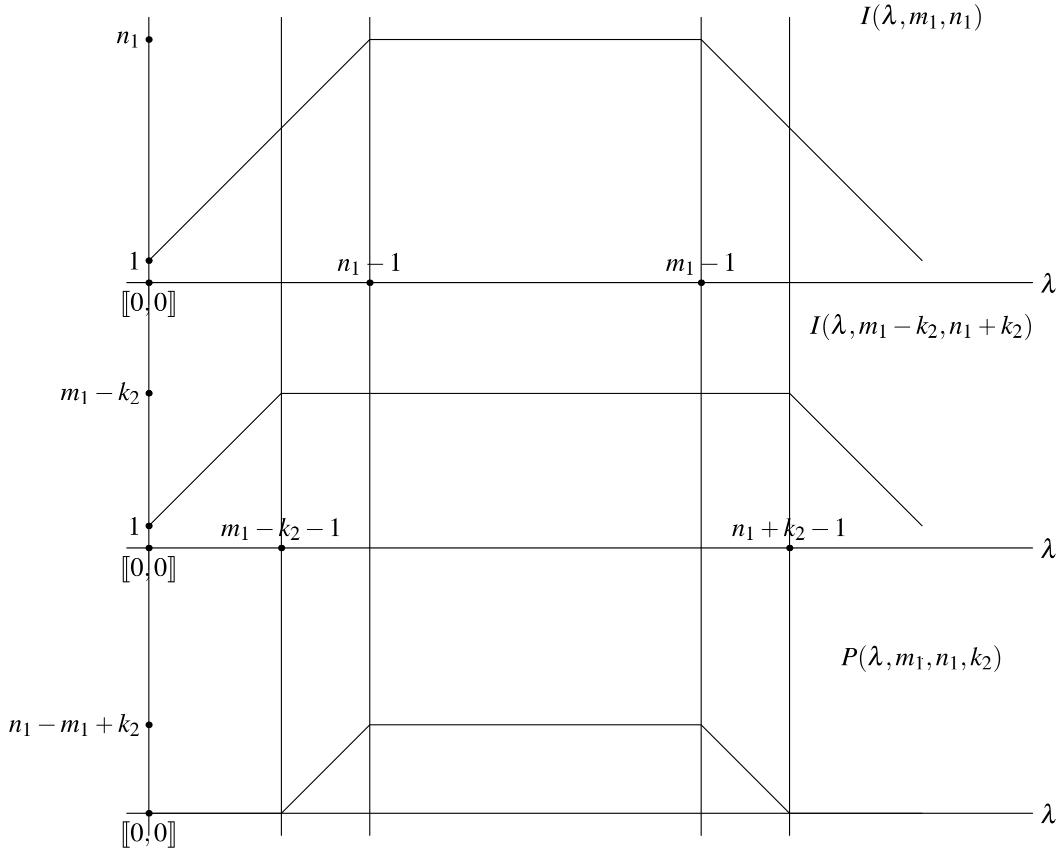


Figure 5: $P(\lambda, m_1, n_1, k_2)$ when $m_1 > n_1$

By Figure 5, we have the following equation:

$$P(\lambda, m_1, n_1, k_2) = \begin{cases} 0 & \text{if } \lambda \in [\![0, m_1 - k_2 - 1]\!], \\ \lambda + 1 - m_1 + k_2 & \text{if } \lambda \in [\![m_1 - k_2 - 1, n_1 - 1]\!], \\ n_1 - m_1 + k_2 & \text{if } \lambda \in [\![n_1 - 1, m_1 - 1]\!], \\ n_1 + k_2 - 1 - \lambda & \text{if } \lambda \in [\![m_1 - 1, n_1 + k_2 - 1]\!], \\ 0 & \text{if } \lambda \in [\![n_1 + k_2 - 1, m_1 + n_1 - 2]\!]. \end{cases} \tag{11.3.11}$$

Furthermore, equation (11.3.11) gives the following equation:

$$\delta(P)(\lambda,m_1,n_1,k_2)=\begin{cases}0 & \text{if } \lambda\in[\![0,m_1-k_2-1]\!],\\ 1 & \text{if } \lambda\in[\![m_1-k_2,n_1-2]\!],\\ n_1+k_2-m_1 & \text{if } \lambda=n_1-1,\\ 0 & \text{if } \lambda\in[\![n_1,m_1-2]\!],\\ n_1+k_2-m_1 & \text{if } \lambda=m_1-1,\\ 1 & \text{if } \lambda\in[\![m_1,n_1+k_2-2]\!],\\ 0 & \text{if } \lambda\in[\![n_1+k_2-1,m_1+n_1-2]\!].\end{cases}$$

This gives inequality (11.3.9) by $n_1+k_2>m_1$ for the fitting μ.

Proof of Claim 2. Claim 2 follows from Claim 1 of Proposition 11.14 and Claim 1. □

11.4 Log-concavity conjectures on some weight-zero parcels

We state log-concavity conjectures on some weight-zero parcels, motivated by Propositions 11.14 and 11.16.

In $\mathbb{Z}_{\geq 0}[q]$, we recall the q-Starling polynomials of the first kind $c(\kappa,\lambda,q)$ and the second kind $S(\kappa,\lambda,q)$. For each $\kappa\in\mathbb{Z}_{\geq 0}$ and $\lambda\in\mathbb{Z}$, the delta function $\delta_{\kappa,\lambda}$ defines

$$c(\kappa,\lambda,q)=\begin{cases}c(\kappa-1,\lambda-1,q)+[\kappa-1]_q c(\kappa-1,\lambda,q) \text{ if } \kappa\geq 1,\\ \delta_{\kappa,\lambda} \text{ if } \kappa=0,\end{cases}$$

$$S(\kappa,\lambda,q)=\begin{cases}S(\kappa-1,\lambda-1,q)+[\lambda]_q S(\kappa-1,\lambda,q) \text{ if } \kappa\geq 1,\\ \delta_{\kappa,\lambda} \text{ if } \kappa=0.\end{cases}$$

The sequences $(c(\kappa,\lambda,q))_{\lambda\in\mathbb{Z}}$ and $(S(\kappa,\lambda,q))_{\lambda\in\mathbb{Z}}$ are strongly q-log-concave [Sag, Theorems 2.4 and 2.5].

Since $c(\kappa,\lambda,q)>_q 0$ and $S(\kappa,\lambda,q)>_q 0$ for $\kappa\geq\lambda\geq 1$, we define the following parcels.

Definition 11.17. *Let $l\in\mathbb{Z}_{\geq 1}$, $\kappa\in\mathbb{Z}_{\geq 1}$, and $s=(1,\kappa)$. Then, let*

$$c_{s,l,q}=\left(c_{s,l,q,m}=\prod_{i\in[\![l]\!]}c(\kappa,m_i,q)\in\mathbb{Z}[q]\right)_{m\in\mathbb{Z}^l},$$

$$S_{s,l,q}=\left(S_{s,l,q,m}=\prod_{i\in[\![l]\!]}S(\kappa,m_i,q)\in\mathbb{Z}[q]\right)_{m\in\mathbb{Z}^l}.$$

Suppose that q is $>_q$-admissible on $\mathfrak{X}$. Then, we call $\Lambda(s,l,>_q,c_{s,l,q},\mathfrak{X})$ and $\Lambda(s,l,>_q,S_{s,l,q},\mathfrak{X})$ q-Stirling parcels of the first and second kinds.

We conjecture the following log-concavity on the merged determinants of q-Stirling parcels.

Conjecture 11.18. *Suppose q-Stirling parcels $\mathscr{F}=\Lambda(s,l,>_q,c_{s,l,q},\mathfrak{X})$ and $\mathscr{G}=\Lambda(s,l,>_q,S_{s,l,q},\mathfrak{X})$. Then, for each fitting (s,l,m,n,k), $\Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X})$ and $\Delta(\mathscr{G})(s,l,m,n,k,\mathfrak{X})$ are log-concave q-polynomials.*

Example 11.19. For $l=1$ and $s=(1,3)$, let $\mathscr{F}=\Lambda(s,l,>_q,c_{s,l,q},\mathfrak{X})$ and $\mathscr{G}=\Lambda(s,l,>_q,S_{s,l,q},\mathfrak{X})$. Suppose $m=(2)$, $n=(1)$, and $k=(0,2)$, which give $m\boxminus k=(0)$ and $n\boxplus k=(3)$. Then, we have the following non-palindromic and log-concave q-polynomials:

$$\Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X})=c(3,2,q)c(3,1,q)=q^2+3q+2;$$
$$\Delta(\mathscr{G})(s,l,m,n,k,\mathfrak{X})=S(3,2,q)S(3,1,q)=q+2.$$

For $\lambda \in \mathbb{Z}_{\geq 0}$, we recall *the Ramanujan and Bessel polynomials* $R_{\lambda+1}(q)$ and $B_\lambda(q)$ in $\mathbb{Z}_{\geq 0}[q]$ such that

$$R_{\lambda+1}(q) = \begin{cases} \lambda(1+q)R_\lambda(q) + q^2 \dfrac{dR_\lambda(q)}{dq} \text{ if } \lambda \in \mathbb{Z}_{\geq 1}, \\ 1 \text{ if } \lambda = 0, \end{cases}$$
$$B_\lambda(q) = \sum_{\kappa \in [\![0,\lambda]\!]} \frac{(\lambda+\kappa)!}{(\lambda-\kappa)!\,\kappa!} \frac{q^\kappa}{2^\kappa}.$$

The sequences $(R_\lambda(q))_{\lambda \in \mathbb{Z}_{\geq 1}}$ and $(B_\lambda(q))_{\lambda \in \mathbb{Z}_{\geq 0}}$ are *strongly q-log-convex* [CWY, Corollaries 3.2 and 3.3].

We define the following parcels by $(R_\lambda(q))_{\lambda \in \mathbb{Z}_{\geq 1}}$ and $(B_\lambda(q))_{\lambda \in \mathbb{Z}_{\geq 0}}$.

Definition 11.20. *Suppose $l \in \mathbb{Z}_{\geq 1}$ and gates $s_1 \geq 1$ and $s_2 \geq 0$. Then, let $R_{s_1,l,q} = \left(R_{s_1,l,q,m} \in \mathbb{Z}[q]\right)_{m \in \mathbb{Z}^l}$ and $B_{s_2,l,q} = \left(B_{s_2,l,q,m} \in \mathbb{Z}[q]\right)_{m \in \mathbb{Z}^l}$ such that*

$$R_{s_1,l,q,m} = \begin{cases} \prod_{i \in [\![l]\!]} R_{m_i}(q) \textit{ for } m \in [\![s_1]\!]^l, \\ 0 \textit{ otherwise}, \end{cases}$$
$$B_{s_2,l,q,m} = \begin{cases} \prod_{i \in [\![l]\!]} B_{m_i}(q) \textit{ for } m \in [\![s_2]\!]^l, \\ 0 \textit{ otherwise}. \end{cases}$$

If q is $>_q$-admissible on $\mathfrak{X}$, then we call $\Lambda(s,l,>_q,R_{s_1,l,q,m},\mathfrak{X})$ and $\Lambda(s,l,>_q,B_{s_2,l,q,m},\mathfrak{X})$ Ramanujan and Bessel parcels.

We conjecture the following log-concavity on the merged determinants of Ramanujan and Bessel parcels.

Conjecture 11.21. *For $s = (1,\infty)$, suppose $\mathscr{F} = \Lambda(s,l,>_q,R_{s,l,q},\mathfrak{X})$ and $\mathscr{G} = \Lambda(s,l,>_q,B_{s,l,q},\mathfrak{X})$. Then, $\Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X})$ and $\Delta(\mathscr{G})(s,l,m,n,k,\mathfrak{X})$ are log-concave q-polynomials for each fitting (s,l,m,n,k).*

Example 11.22. For $l = 1$ and $s = (1,\infty)$, let $\mathscr{F} = \Lambda(s,l,>_q,R_{s,l,q},\mathfrak{X})$ and $\mathscr{G} = \Lambda(s,l,>_q,B_{s,l,q},\mathfrak{X})$. If $m = (3)$, $n = (2)$, and $k = (0,2)$, then $m \boxminus k = (1)$ and $n \boxplus k = (4)$. Furthermore, we have the following log-concave q-polynomials:

$$-\Delta(\mathscr{F})(s,l,m,n,k,\mathfrak{X}) = -(R(3,q)R(2,q) - R(1,q)R(4,q)) = 12q^3 + 18q^2 + 12q + 4;$$
$$-\Delta(\mathscr{G})(s,l,m,n,k,\mathfrak{X}) = -(B(3,q)B(2,q) - B(1,q)B(4,q)) = 60q^5 + 120q^4 + 72q^3 + 19q^2 + 2q.$$

12 Almost strictly unimodal sequences and Young diagrams

We introduce the notion of fitting paths to connect fitting tuples. Also, we introduce the notion of flip-invariant parcels. Then, we obtain almost strictly unimodal sequences along fitting paths by the merged-log-concavity of flip-invariant parcels. In particular, Young diagrams give infinite-length almost strictly unimodal sequences.

12.1 Strict log-concavity and merged-log-concavity

The strict log-concavity of positive real numbers implies the almost strict unimodality. But, this implication does not always extend to the merged-log-concavity of parcels. A reason is that $\boxplus$ and $\boxminus$ do not cancel as $+$ and $-$. Let us explain the distinction of the log-concave notions on positive real numbers and parcels with more detail.

Suppose a strictly log-concave sequence $r = (r_i \in \mathbb{R}_{>0})_{i \in [\![s]\!]}$. Then, $j-1, j+2 \in [\![s]\!]$ give

$$
\begin{aligned}
r_j^2 - r_{j-1} r_{j+1} &> 0, \\
r_{j+1}^2 - r_j r_{j+2} &> 0.
\end{aligned}
\tag{12.1.1}
$$

Furthermore, r is almost strictly unimodal by $\frac{r_j}{r_{j-1}} > \frac{r_{j+1}}{r_j} > \frac{r_{j+2}}{r_{j+1}}$. However, inequality (12.1.1) does not always extend to merged-log-concave parcels.

For instance, consider a $\succ'$-merged-log-concave $\mathscr{F} = \Lambda(s, l, w, \succ, f_s, \phi, \rho, x, \mathfrak{X})$. Then, fitting (s, l, m, m, k) and $(s, l, m \boxplus k, m \boxplus k, k)$ yield

$$
\begin{aligned}
\Upsilon(s, l, w, m, m, k, \phi, \rho, x, \mathfrak{X})(\mathscr{F}_m \mathscr{F}_{m^\vee} - \mathscr{F}_{m \boxminus k} \mathscr{F}_{(m \boxplus k)^\vee}) &\succ' 0, \\
\Upsilon(s, l, w, m \boxplus k, m \boxplus k, k, \phi, \rho, x, \mathfrak{X})(\mathscr{F}_{m \boxplus k} \mathscr{F}_{(m \boxplus k)^\vee} - \mathscr{F}_{(m \boxplus k) \boxminus k} \mathscr{F}_{((m \boxplus k) \boxplus k)^\vee}) &\succ' 0.
\end{aligned}
$$

However, in general, we do not have $\mathscr{F}_{(m \boxplus k) \boxminus k} = \mathscr{F}_m$, which corresponds to

$$
r_{(j+1)-1} = r_j \tag{12.1.2}
$$

in inequality (12.1.1).

Example 12.1. Suppose $s = (0, \infty)$, $l = 2$, $m = (0, 1)$, and $k = (0, 0, 2, 1)$. Then, (s, l, m, m, k) and $(s, l, m \boxplus k, m \boxplus k, k)$ are fitting by $\nu(k) = (0, 0, 2, 3)$ and $\nu(m, m, k) = (0, 1, 2, 4)$. But, $\sigma(k) = (3, 2)$ implies

$$
(m \boxplus k) \boxminus k = (m + \sigma(k)^\vee) - \sigma(k) = (0, 1) \neq m.
$$

12.2 Fitting paths

Suppose a fitting (s, l, m, m, k). We achieve $(m \boxplus k) \boxminus k = m$ if and only if

$$
\sigma(k) = \sigma(k)^\vee. \tag{12.2.1}
$$

But, equation (12.2.1) forces

$$
k[2 : 2l] = \iota^{l-1}(0) \mathbin{+\!\!+} (\lambda) \mathbin{+\!\!+} \iota^{l-1}(0) \tag{12.2.2}
$$

for some $\lambda \in \mathbb{Z}_{\geq 1}$, since $k_i > 0$ of $i \neq l+1, 1$ makes $\sigma(k)_l < \sigma(k)_l + k_i \leq \sigma(k)_1$. To establish an analog of equation (12.1.2) for more general k, we define the following notion of fitting paths.

Definition 12.2. *Suppose a width-two gate θ. Consider a sequence $P = (P_i = (s, l, m_i, n_i, k_i))_{i \in [\![\theta]\!]}$ such that P_{θ_1} and P_{θ_2} are fitting tuples, k_{θ_1} and k_{θ_2} are σ-equivalent, and $m_{\theta_2} = (n_{\theta_1} \boxplus k_{\theta_1})^\vee$.*

1. *For m_{θ_1} and n_{θ_1}, assume the equation $m_{\theta_1} = n_{\theta_1}$.*
   (a) *We call P a fitting path of type e-e if $m_{\theta_2} = n_{\theta_2}$.*
   (b) *We call P a fitting path of type e-f if $m_{\theta_2} = n_{\theta_2}^\vee$.*
2. *For m_{θ_1} and n_{θ_1}, assume the flipped equation $m_{\theta_1}^\vee = n_{\theta_1}$.*
   (a) *We call P a fitting path of type f-e if $m_{\theta_2} = n_{\theta_2}$.*
   (b) *We call P a fitting path of type f-f if $m_{\theta_2} = n_{\theta_2}^\vee$.*

3. *We call P a fitting path P of type free if $m_i = n_i = m_i^\vee$ for each $i \in [\![\theta]\!]$.*

We define the notion of fitting paths of general lengths.

Definition 12.3. *Let θ be a gate. Consider a sequence $P = (P_i = (s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$ such that $Q_j = (P_i)_{i\in[\![j,j+1]\!]}$ is a fitting path for each $j \in [\![\theta_1, \theta_2 - 1]\!]$.*

1. *We call P a fitting path.*

2. *We call P a fitting path of type A if Q_j has the same type A for all $j \in [\![\theta_1, \theta_2 - 1]\!]$.*

We refer to s, l, θ, and $(k_{i,1})_{i\in[\![\theta]\!]}$ of P as the gate, width, length, free-path parameters of P.

If θ is of infinite-width, then we say that P is of infinite-length. If θ is of width-n and $n \geq 2$, then we say that P is of length-n. If θ is of width-one and P_{θ_1} is fitting, then we call P a fitting path of length-one.

On the terminology of the free-path parameters of P, see Remark 3.5.

Example 12.4. Let $s = (0,\infty)$, $l = 1$, $m_1 = (1)$, and $k_1 = (0,1)$. Then, $\mu_1 = (s,l,m_1,m_1,k_1)$ is fitting by

$$\begin{aligned} \nu(k_1) &= (0,1)\,, \\ \nu(m_1,m_1,k_1) &= \nu(k_1) + m_1 +\!\!\!+\, m_1 = (1,2)\,. \end{aligned}$$

Also, let $m_2 = (m_1 \boxplus k_1)^\vee = (2)$ and $k_2 = k_1$. Then, $\mu_2 = (s,l,m_2,m_2,k_2)$ is fitting by

$$\nu(m_2,m_2,k_2) = \nu(k_2) + m_2 +\!\!\!+\, m_2 = (2,3)\,.$$

By $l = 1$, we deduce that $(\mu_i)_{i\in[\![1,2]\!]}$ is a fitting path of type free.

Example 12.5. We have the following fitting path with different but equivalent supports. Let $s = (0,\infty)$, $l = 2$, $m_1 = (0,3)$, and $k_1 = (0,4,4,4)$. Then, $\mu_1 = (s,l,m_1,m_1^\vee,k_1)$ is fitting, since

$$\begin{aligned} \nu(k_1) &= (0,4,8,12)\,, \\ \nu(m_1,n_1,k_1) &= \nu(k_1) + m_1 +\!\!\!+\, m_1^\vee = (0,7,11,12)\,. \end{aligned}$$

If we were to continue with k_1, either $(s,l,(m_1^\vee \boxplus k_1)^\vee, m_1^\vee \boxplus k_1, k_1)$ or $(s,l,(m_1^\vee \boxplus k_1)^\vee, (m_1^\vee \boxplus k_1)^\vee, k_1)$ has to be fitting. However, neither of them is fitting, since $m_1^\vee \boxplus k_1 = (3,0) + (4,12) = (7,12)$ implies

$$\begin{aligned} \nu((m_1^\vee \boxplus k_1)^\vee, m_1^\vee \boxplus k_1, k_1) &= \nu(k_1) + (12,7) +\!\!\!+\, (7,12) = (12,11,15,24)\,, \\ \nu((m_1^\vee \boxplus k_1)^\vee, (m_1^\vee \boxplus k_1)^\vee, k_1) &= \nu(k_1) + (12,7) +\!\!\!+\, (12,7) = (12,11,20,19)\,. \end{aligned}$$

Instead, let $k_2 = (0,7,4,1) \neq k_1$. Not only are k_1 and k_2 equivalent, but also $\mu_2 = (s,l,m_2,n_2,k_2) = (s,l,(m_1^\vee \boxplus k_1)^\vee, m_1 \boxplus k_1, k_2)$ is fitting, since

$$\begin{aligned} \nu(k_2) &= (0,7,11,12)\,, \\ \nu(m_2,n_2,k_2) &= \nu(k_2) + (12,7) +\!\!\!+\, (7,12) = (12,14,18,24)\,. \end{aligned}$$

In particular, $(\mu_i)_{i\in[\![2]\!]}$ is a fitting path of type f-f with different but equivalent k_1 and k_2.

Since we do not demand equation (12.2.1) on the fitting tuples of fitting paths, we introduce the following notion of parcels to obtain an analog of equation (12.1.2).

Definition 12.6. *Let $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$. We call $\mathscr{F}$ flip-invariant at $r \in \mathrm{O}_{\mathfrak{X}}$ if each $m \in \mathbb{Z}^l$ satisfies*

$$\mathscr{F}_m(r) = \mathscr{F}_{m^\vee}(r).$$

We simply call $\mathscr{F}$ flip-invariant if $\mathscr{F}$ is flip-invariant at every $r \in \mathrm{O}_{\mathfrak{X}}$.

We have the following flip-invariant parcels.

Lemma 12.7. *Suppose $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ and $r \in \mathrm{O}_{\mathfrak{X}}$ such that x, $\phi(x)$, and w are palindromic tuples, and $f_{s,m}(r) = f_{s,m^\vee}(r)$ for each $m \in \mathbb{Z}^l$. Then, $\mathscr{F}$ is flip-invariant at r.*

Proof. Since $\mathscr{F}_m = \frac{f_m}{\prod \phi(x)^{m \circ w} \cdot [m]!_x^w}$ for each $m \in [\![s]\!]^l$, the statement follows. □

In particular, the following monomial parcels are flip-invariant.

Proposition 12.8. *Suppose a monomial index (l,w,γ) with palindromic w, $(\gamma_{i,1})_{i\in[\![l]\!]}$, and $(\gamma_{i,2})_{i\in[\![l]\!]}$. Then, a monomial parcel $\Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\rho,x,\mathfrak{X})$ is flip-invariant.*

Proof. The statement follows from Lemma 12.7. □

For further discussion, we state the following equations on the σ-plus and σ-minus.

Lemma 12.9. *Let $l \in \mathbb{Z}_{\geq 1}$, $m \in \mathbb{Z}^l$, and $k,k' \in \mathbb{Z}^{2l}$. Then, we have the following equations.*

$$(m \boxplus k)^\vee = m^\vee \boxminus (-k), \tag{12.2.3}$$
$$(m \boxminus k)^\vee = m^\vee \boxplus (-k), \tag{12.2.4}$$
$$m \boxplus k \boxplus k' = m \boxplus (k+k'), \tag{12.2.5}$$
$$m \boxminus k \boxminus k' = m \boxminus (k+k'), \tag{12.2.6}$$
$$(m \boxplus k)^\vee \boxminus k = m^\vee, \tag{12.2.7}$$
$$(m \boxminus k)^\vee \boxplus k = m^\vee, \tag{12.2.8}$$
$$m \boxminus k \boxplus k' = m \boxplus k' \boxminus k. \tag{12.2.9}$$

Proof. First, we obtain equation (12.2.3) by

$$(m \boxplus k)^\vee = m^\vee + (\sigma(k)^\vee)^\vee = m^\vee - \sigma(-k) = m^\vee \boxminus (-k).$$

Replacing m and k with $m^\vee$ and $-k$, we deduce equation (12.2.4) from equation (12.2.3).

Second, we obtain equation (12.2.5) by

$$m \boxplus k \boxplus k' = m + \sigma(k)^\vee + \sigma(k')^\vee = m + \sigma(k+k')^\vee = m \boxplus (k+k').$$

Similarly, equation (12.2.6) follows from $\sigma(k) + \sigma(k') = \sigma(k+k')$.

Third, we have equation (12.2.7), since equations (12.2.3) and (12.2.6) give

$$(m \boxplus k)^\vee \boxminus k = m^\vee \boxminus (-k) \boxminus k = m^\vee.$$

Similarly, equation (12.2.8) holds by equations (12.2.4) and (12.2.5).

Finally, we have equation (12.2.9), since

$$m \boxminus k \boxplus k' = m - \sigma(k) + \sigma(k')^\vee = m + \sigma(k')^\vee - \sigma(k) = m \boxplus k' \boxminus k.$$

□

We derive the following analog of equation (12.1.2) by fitting paths and flip-invariant parcels.

Proposition 12.10. *Consider a fitting path $P = (P_i = (s,l,m_i,n_i,k_i))_{i \in [\![\theta]\!]}$ and flip-invariant parcel $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ at $r \in \mathrm{O}_{\mathfrak{X}}$. Then, each $j \in [\![\theta_1+1,\theta_2]\!]$ satisfies*

$$\mathscr{F}_{m_j \boxminus k_j}(r) = \mathscr{F}_{n_{j-1}}(r) = \mathscr{F}_{m_{j-1}}(r). \tag{12.2.10}$$

Proof. First, we have $m_j = (n_{j-1} \boxplus k_{j-1})^{\vee}$ or $m_j = (n_{j-1}^{\vee} \boxplus k_{j-1})^{\vee}$, since P is a fitting path. Then, equations 12.2.3 and 12.2.6 of Lemma 12.9 imply $m_j \boxminus k_j = n_{j-1}$ or $n_{j-1}^{\vee}$. This equation gives the left-hand side of equation (12.2.10) by the flip-invariance of $\mathscr{F}$.

Second, we have $m_{j-1} = n_{j-1}$ or $n_{j-1}^{\vee}$ for the fitting path P. The right-hand side of equation (12.2.10) holds again by the flip-invariance of $\mathscr{F}$. □

12.3 Positivity of ring shift factors

We generalize inequality (1.6.2) by the ring shift factors in Definition 6.1. We state the following lemma to obtain the generalization by mediators and q-numbers, which factorize the ring shift factors.

Lemma 12.11. *The set $C = \left\{ f \in \mathbb{Q}(\mathfrak{X}) \mid f >_{\mathrm{O}_{\mathfrak{X}}} 0 \right\}$ is a group under multiplications.*

Proof. Clearly, $1 \in C$. If $g \in C^2$, then $\mathrm{Frac}(g)(r) = \mathrm{Frac}(g(r)) > 0$ for each $r \in \mathrm{O}_{\mathfrak{X}}$. □

We state the following $>_{\mathrm{O}_{\mathfrak{X}}}$-positivities of mediators and q-numbers.

Proposition 12.12. *Suppose a $\mu = (s,l,w,\succ,\rho,x,\mathfrak{X})$-mediator ϕ.*

1. *If $m \in \mathbb{Z}^l_{\geq 0}$ and $\lambda \in \mathbb{Z}^l_{\geq 1}$, then*

$$\prod \phi(x)^{m \circ w} >_{\mathrm{O}_{\mathfrak{X}}} 0, \tag{12.3.1}$$

$$[m]!^{w}_{x^{\lambda}} >_{\mathrm{O}_{\mathfrak{X}}} 0. \tag{12.3.2}$$

2. *If $m \in [\![s]\!]^l$, then*

$$\prod \phi(x^{\rho})^{m \circ w} >_{\mathrm{O}_{\mathfrak{X}}} 0.$$

3. *Suppose a fitting (s,l,m,n,k) with $a = \nu(k)$ and $b = \nu(m,n,k)$. Then, it follows that*

$$\prod (\phi(x^{\rho})^{\sqcup})^{(b-a)\circ w^{\sqcup}} >_{\mathrm{O}_{\mathfrak{X}}} 0.$$

Proof. Proof of Claim 1. First, the base positivity of ϕ and μ implies inequality (12.3.1) by Lemma 12.11.

Second, we prove inequality (12.3.2). By $\lambda \geq 1$, each $i \in [\![l]\!]$ implies $[m_i]!^{w_i}_{x_i^{\lambda_i}} >_{x_i} 0$. This gives $[m]!^{w}_{x^{\lambda}} \succ 0$ by the half $>_x$-$\succ$ implication in Claim 2 of Lemma 5.20. By the half $\succ$-$>_{\mathrm{O}_{\mathfrak{X}}}$ implication, we deduce inequality (12.3.2).

Proof of Claim 2. By the compatibility $>_{\mathrm{O}_{\mathfrak{X}}} \Supset \succ$, the base-shift positivity of ϕ and μ implies

$$B(s,l,w,m,\phi,\rho,x,\mathfrak{X}) >_{\mathrm{O}_{\mathfrak{X}}} 0.$$

Then, we have

$$B(s,l,w,m,\phi,\rho,x,\mathfrak{X}) = \prod_{i\in[\![l]\!]} \frac{\phi(x_i^{\rho_i})^{w_i m_i}[m_i]!^{w_i}_{x_i^{\rho_i}}}{\phi(x)^{w_i m_i}[m_i]!^{w_i}_{x_i}} = \frac{\prod \phi(x^{\rho})^{m\circ w}\cdot [m]!^{w}_{x^{\rho}}}{\prod \phi(x)^{m\circ w}\cdot [m]!^{w}_{x}} >_{\mathrm{O}_{\mathfrak{X}}} 0.$$

Therefore, Claim 2 follows from Lemma 12.11 and Claim 1.

Proof of Claim 3. Claim 2 gives Claim 3, since $m, n^{\vee} \in [\![s]\!]^l$ satisfy

$$\prod (\phi(x^{\rho})^{\sqcup})^{(b-a)\circ w^{\sqcup}} = \prod (\phi(x^{\rho})^{\sqcup})^{(m+\!\!\!+n)\circ w^{\sqcup}} = \prod \phi(x^{\rho})^{m\circ w}\cdot \prod \phi(x^{\rho})^{n^{\vee}\circ w}.$$

□

We obtain the generalization of inequality (1.6.2) by the ring shift factors.

Corollary 12.13. *Suppose a fitting $\mu = (s,l,m,n,k)$. Then,*

$$\Upsilon(s,l,w,m,n,k,\phi,\rho,x,\mathfrak{X}) >_{\mathrm{O}_{\mathfrak{X}}} 0.$$

Proof. Let $a = \nu(k)$ and $b = \nu(m,n,k)$. By the slope conditions of μ, we have $a, b \geq 0$. We deduce the assertion by Item 2 of Definition 6.1, Lemma 12.11, and Claims 1 and 3 of Proposition 12.12. □

12.4 Almost strictly unimodal sequences by the merged-log-concavity

We discuss almost strictly unimodal sequences by fitting paths and the merged-log-concavity. We introduce the following notation along fitting paths.

Definition 12.14. *Let θ be a gate. Consider the pair $\zeta = (P, \mathscr{F})$ of a fitting path $P = ((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$ and parcel $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$. Let $r \in \mathrm{O}_{\mathfrak{X}}$.*

1. *We define the extended gate $e(\theta) = (\theta_1 - 1, \theta_2 + 1) \in \hat{\mathbb{Z}}^2$.*
2. *We define the path-parcel sequence $u(\zeta, r) = (u(\zeta,r)_i \in \mathbb{R})_{i\in[\![e(\theta)]\!]}$ such that*

$$u(\zeta,r)_i = \begin{cases} \mathscr{F}_{m_{\theta_1}\boxminus k_{\theta_1}}(r) & \text{if } i = \theta_1 - 1, \\ \mathscr{F}_{n_i}(r) & \text{if } i \in [\![\theta_1, \theta_2]\!], \\ \mathscr{F}_{n_{\theta_2}\boxplus k_{\theta_2}}(r) & \text{if } i = \theta_2 + 1 < \infty. \end{cases}$$

By the ends of parcel-path sequences, we define the notion of wrapped fitting paths.

Definition 12.15. *Suppose a fitting path $P = ((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$. We call P wrapped if P satisfies the following conditions:*

$$m_{\theta_1} \boxminus k_{\theta_1} \in [\![s]\!]^l; \tag{12.4.1}$$

$$n_{\theta_2} \boxplus k_{\theta_2} \in [\![s]\!]^l \text{ when } \theta_2 < \infty. \tag{12.4.2}$$

We refer to (12.4.1) *and* (12.4.2) *as the lower inclusion condition and the upper inclusion condition of P.*

We state the following lemma for a later reference.

Lemma 12.16. *If $r = (r_i \in \mathbb{R}_{>0})_{i \in [\![s]\!]}$ is strictly log-concave, then r is almost strictly unimodal.*

Proof. Let $j-1, j+1 \in [\![s]\!]$. The strict log-concavity of r reads $r_j^2 - r_{j-1}r_{j+1} > 0$. Then, by $r_{j-1}, r_j > 0$, we have $\frac{r_j}{r_{j-1}} > \frac{r_{j+1}}{r_j}$. Since $\frac{r_j}{r_{j-1}} \leq 1$ implies $r_{j-1} \geq r_j > r_{j+1} > \dots$, the assertion follows. $\square$

Since each $\succ$-merged-log-concavity gives the $>_{O_{\mathfrak{X}}}$-merged-log-concavity by the half $\succ$-$>_{O_{\mathfrak{X}}}$ implication, we state the following short almost strictly unimodal sequences by the $>_{O_{\mathfrak{X}}}$-merged-log-concavity.

Lemma 12.17. *Consider a length-one fitting path $P = ((s,l,m_i,n_i,k_i))_{i \in [\![\theta]\!]}$ and $r \in O_{\mathfrak{X}}$. Let $\mathscr{F} = \Lambda(s,l,w,\succ, f_s,\phi,\rho,x,\mathfrak{X})$ be $>_{O_{\mathfrak{X}}}$-merged-log-concave and flip-invariant at r. Let $\zeta = (P,\mathscr{F})$. Then, we have the following.*

1. *$u(\zeta,r)$ is strictly log-concave.*
2. *$u(\zeta,r) > 0$ is almost strictly unimodal if P is wrapped.*

Proof. Proof of Claim 1. Suppose $\theta = (1,1)$ without loss of generality. By $e(\theta) = (0,2)$, we want to prove the strict log-concavity of the following sequence:

$$u(\zeta,r)_0 = \mathscr{F}_{m_1 \boxminus k_1}(r); \tag{12.4.3}$$
$$u(\zeta,r)_1 = \mathscr{F}_{n_1}(r); \tag{12.4.4}$$
$$u(\zeta,r)_2 = \mathscr{F}_{n_1 \boxplus k_1}(r). \tag{12.4.5}$$

By the $>_{O_{\mathfrak{X}}}$-merged-log-concavity of $\mathscr{F}$, we have

$$\Delta(\mathscr{F})(s,l,w,m_1,n_1,k_1,\phi,\rho,x,\mathfrak{X}) = \Upsilon(s,l,w,m_1,n_1,k_1,\phi,\rho,x,\mathfrak{X}) \cdot \det(\mathscr{F},m_1,n_1,k_1) >_{O_{\mathfrak{X}}} 0$$

By Lemma 12.11 and Corollary 12.13, we deduce

$$\det(\mathscr{F},m_1,n_1,k_1) = \mathscr{F}_{m_1}\mathscr{F}_{n_1^\vee} - \mathscr{F}_{m_1 \boxminus k_1}\mathscr{F}_{(n_1 \boxplus k_1)^\vee} >_{O_{\mathfrak{X}}} 0.$$

Also, $m_1 = n_1^\vee$ or $m_1 = n_1$ for the fitting path P. Since $\mathscr{F}$ is flip-invariant at r, we obtain

$$\mathscr{F}_{n_1}(r)\mathscr{F}_{n_1}(r) - \mathscr{F}_{m_1 \boxminus k_1}(r)\mathscr{F}_{n_1 \boxplus k_1}(r) >_{O_{\mathfrak{X}}} 0.$$

Claim 1 follows, since $u(\zeta,r)_1^2 - u(\zeta,r)_0 u(\zeta,r)_2 > 0$ by equations (12.4.3), (12.4.4), and (12.4.5).

Proof of Claim 2. Claim 2 holds by Lemma 12.16 and Claim 1, because $m_1 \boxminus k_1 \in [\![s]\!]^l$ and $n_1 \boxplus k_1 \in [\![s]\!]^l$ imply $u(\zeta,r)_0, u(\zeta,r)_2 > 0$ by equations (12.4.3) and (12.4.5). $\square$

Moreover, we obtain the following almost strictly unimodal sequences by general fitting paths.

Theorem 12.18. *Consider a fitting path $P = (P_i = (s,l,m_i,n_i,k_i))_{i \in [\![\theta]\!]}$. Assume that $\mathscr{F} = \Lambda(s,l,w,\succ, f_s,\phi,\rho,x,\mathfrak{X})$ is $>_{O_{\mathfrak{X}}}$-merged-log-concave and flip-invariant at $r \in O_{\mathfrak{X}}$. Let $\zeta = (P,\mathscr{F})$. Then, we have the following.*

1. *$u(\zeta,r)$ is strictly log-concave.*
2. *$u(\zeta,r) > 0$ is almost strictly unimodal if P is wrapped.*

Proof. Proof of Claim 1. For each $\lambda \in [\![\theta]\!]$, consider the length-one fitting path $Q_\lambda = (P_i)_{i \in [\![\lambda,\lambda]\!]}$. Let $\kappa_\lambda = (Q_\lambda, \mathscr{F})$. Then, we prove the following equations:

$$u(\kappa_\lambda, r)_{\lambda-1} = u(\zeta, r)_{\lambda-1}; \tag{12.4.6}$$
$$u(\kappa_\lambda, r)_{\lambda} = u(\zeta, r)_{\lambda}; \tag{12.4.7}$$
$$u(\kappa_\lambda, r)_{\lambda+1} = u(\zeta, r)_{\lambda+1}. \tag{12.4.8}$$

These equations imply Claim 1 by Claim 1 of Lemma 12.17 on $u(\kappa_\lambda, r)$.

First, we prove equation (12.4.6). If $\lambda - 1 \notin [\![\theta]\!]$, then

$$u(\kappa_\lambda, r)_{\lambda-1} = \mathscr{F}_{m_\lambda \boxminus k_\lambda}(r) = u(\zeta, r)_{\lambda-1}.$$

If $\lambda - 1 \in [\![\theta]\!]$, then Proposition 12.10 gives

$$u(\kappa_\lambda, r)_{\lambda-1} = \mathscr{F}_{m_\lambda \boxminus k_\lambda}(r) = \mathscr{F}_{n_{\lambda-1}}(r) = u(\zeta, r)_{\lambda-1}.$$

Second, equation (12.4.7) holds, since $\lambda \in [\![\theta]\!]$ implies

$$u(\kappa_\lambda, r)_{\lambda} = \mathscr{F}_{n_\lambda}(r) = u(\zeta, r)_{\lambda}.$$

Third, we prove equation (12.4.8). If $\lambda + 1 \notin [\![\theta]\!]$, then equation (12.4.8) follows from

$$u(\kappa_\lambda, r)_{\lambda+1} = \mathscr{F}_{n_\lambda \boxplus k_\lambda}(r) = u(\zeta, r)_{\lambda+1}.$$

If $\lambda + 1 \in [\![\theta]\!]$, then $m_{\lambda+1} = (n_\lambda \boxplus k_\lambda)^\vee$, which is $n_{\lambda+1}$ or $n_{\lambda+1}^\vee$. By the flip-invariance of $\mathscr{F}$, we deduce

$$u(\kappa_\lambda, r)_{\lambda+1} = \mathscr{F}_{n_\lambda \boxplus k_\lambda}(r) = \mathscr{F}_{n_{\lambda+1}}(r) = u(\zeta, r)_{\lambda+1}.$$

Proof of Claim 2. Since P is wrapped, we have

$$u(\zeta, r)_{\theta_1 - 1} = \mathscr{F}_{m_{\theta_1} \boxminus k_{\theta_1}}(r) > 0,$$
$$u(\zeta, r)_{\theta_2 + 1} = \mathscr{F}_{n_{\theta_2} \boxplus k_{\theta_2}}(r) > 0 \text{ if } \theta_2 < \infty.$$

Claim 2 follows from Lemma 12.16. □

12.5 Infinite-length fitting paths

We introduce the following sequences to obtain infinite-length fitting paths, which in turn give infinite-length almost strictly unimodal sequences in Theorem 12.18.

Definition 12.19. *Let $l \in \mathbb{Z}_{\geq 1}$. Let $\lambda \in \mathbb{Z}^3_{\geq 0}$ such that $l \geq \lambda_1 \geq 1$ and $\lambda_2 \geq 1$. Suppose infinite gates $s \geq 0$ and $\theta = (1, \infty) \in \hat{\mathbb{Z}}^2$. Let $t = (t_i \in \mathbb{Z}_{\geq 0})_{i \in [\![\theta]\!]}$. Then, we define the sequence*

$$P_{s,l,\lambda,t} = \left(P_{s,l,\lambda,t,i} = (s, l, m_i, n_i, k_i)\right)_{i \in [\![\theta]\!]}$$

for $m_i, n_i \in \mathbb{Z}^l_{\geq 0}$ and $k_i \in \mathbb{Z}^{2l}_{\geq 0}$ as follows.

1. *If $\lambda_1 = l$, then*

$$m_i = n_i = \iota^l(\lambda_2 + \lambda_3 + s_1) + (i-1)\iota^l(\lambda_2),$$
$$k_i = \iota^l(0) + (\lambda_2) + \iota^{l-1}(0) + (t_i) + \iota^{2l-1}(0).$$

2. *If $\lambda_1 < l$, then*

$$\begin{aligned} m_i = n_i &= \iota^{\lambda_1}(\lambda_2) + \iota^{l-\lambda_1}(0) + (i-1)\iota^l(2\lambda_2) + \iota^l(\lambda_3 + s_1), \\ k_i &= \iota^{\lambda_1}(0) + (\lambda_2) + \iota^{l-\lambda_1-1}(0) + (\lambda_2) + \iota^{\lambda_1-1}(0) + (\lambda_2) + \iota^{l-\lambda_1-1}(0) + (t_i) + \iota^{2l-1}(0). \end{aligned}$$

When $t = \iota^{[\![\theta]\!]}(0) = (t_i = 0)_{i \in [\![\theta]\!]}$, let $P_{s,l,\lambda} = P_{s,l,\lambda,t}$ for simplicity.

We prove that $P_{s,l,\lambda,t}$ is an infinite-length fitting path. In particular, for each $P_{s,l,\lambda,t,i}$, we verify the slope conditions, which allow non-strict inequalities on $\nu(k_i)$ and $\nu(m_i, n_i, k_i)$.

For Item 1 of Definition 12.19, we have the following.

Proposition 12.20. *Assume $l \in \mathbb{Z}_{\geq 1}$ and $\lambda \in \mathbb{Z}^3$ such that*

$$\lambda_1 = l,\ \lambda_2 \geq 1,\ \lambda_3 \geq 0. \tag{12.5.1}$$

Then, $P_{s,l,\lambda,t} = \left(P_{s,l,\lambda,t,i} = (s, l, m_i, n_i, k_i)\right)_{i \in [\![\theta]\!]}$ is an infinite-length fitting path of type free such that

$$\nu(k_i) = \iota^l(0) + \iota^l(\lambda_2) + t_i. \tag{12.5.2}$$

Proof. For simplicity, let $t = \iota^{[\![\theta]\!]}(0)$. First, we prove that each $P_{s,l,\lambda,i}$ is fitting. We have $m_1, n_1 \geq s_1$ by condition (12.5.1). This gives $m_i, n_i \in [\![s]\!]^l$, since s is an infinite gate. Also, we obtain equation (12.5.2), since

$$k_i = \iota^l(0) + (\lambda_2) + \iota^{l-1}(0).$$

This gives

$$\nu(m_i, n_i, k_i) = \iota^l(\lambda_2 + \lambda_3 + s_1) + \iota^l(2\lambda_2 + \lambda_3 + s_1) + (i-1)\iota^{2l}(\lambda_2). \tag{12.5.3}$$

Each $P_{s,l,\lambda,i}$ is fitting by equations (12.5.2) and (12.5.3), since condition (12.5.1) implies the slope conditions on $\nu(k_i)$ and $\nu(m_i, n_i, k_i)$.

Second, we prove that $Q_i = \left(Q_{i,\kappa} = P_{s,l,\lambda,i+\kappa}\right)_{\kappa \in [\![0,1]\!]}$ is a fitting path of type free for each $i \in [\![\theta]\!]$. Since $\sigma(k_i) = \iota^l(\lambda_2)$, we have

$$(n_i \boxplus k_i)^\vee = \iota^l(\lambda_2 + \lambda_3 + s_1) + i\iota^l(\lambda_2) = m_{i+1}.$$

We deduce that each Q_i is a fitting path of type free, since $m_{i+\kappa} = n_{i+\kappa} = m_{i+\kappa}^\vee$ for $\kappa \in [\![0,1]\!]$. □

Example 12.21. Let $s = (0, \infty)$, $l \in \mathbb{Z}_{\geq 1}$, and $\lambda = (l, 1, 0)$. Then, $\theta = (1, \infty)$ gives the infinite-length fitting path $P_{s,l,\lambda} = \left(P_{s,l,\lambda,i} = (s, l, m_i, n_i, k_i)\right)_{i \in [\![\theta]\!]}$ of type free such that each $i \in [\![\theta]\!]$ satisfies

$$\begin{aligned} m_i = n_i &= \iota^l(i), \\ k_i &= \iota^l(0) + (1) + \iota^{l-1}(0). \end{aligned}$$

In particular, this appears in equation (12.2.2).

For Item 2 of Definition 12.19, we have the following.

Proposition 12.22. *Suppose $l \in \mathbb{Z}_{\geq 1}$ and $\lambda \in \mathbb{Z}^3$ such that*

$$1 \leq \lambda_1 < l,\ \lambda_2 \geq 1,\ \lambda_3 \geq 0. \tag{12.5.4}$$

Consider $P_{s,l,\lambda,t} = \left(P_{s,l,\lambda,t,i} = (s,l,m_i,n_i,k_i)\right)_{i \in [\![\theta]\!]}$. Also, let

$$L(l,\lambda_1) = \min(\lambda_1 - 1, l - \lambda_1 - 1),$$
$$H(l,\lambda_1) = \max(\lambda_1 - 1, l - \lambda_1 - 1).$$

Then, the following statements hold.

1. *We have*

$$\sigma(k_i)_{l-j} = \begin{cases} \lambda_2 & \text{if } j \in [\![0, L(l,\lambda_1)]\!], \\ 2\lambda_2 & \text{if } j \in [\![L(l,\lambda_1)+1, H(l,\lambda_1)]\!], \\ 3\lambda_2 & \text{if } j \in [\![H(l,\lambda_1)+1, l-1]\!]. \end{cases}$$

2. *$P_{s,l,\lambda,t}$ is an infinite-length fitting path of type e-e such that*

$$\nu(k_i) = \iota^{\lambda_1}(0) \mathbin{+\!\!+} \iota^{l-\lambda_1}(\lambda_2) \mathbin{+\!\!+} \iota^{\lambda_1}(2\lambda_2) \mathbin{+\!\!+} \iota^{l-\lambda_1}(3\lambda_2) + t_i \tag{12.5.5}$$

for each $i \in [\![\theta]\!]$.

Proof. For simplicity, let $t = \iota^{[\![\theta]\!]}(0)$.

Proof of Claim 1. If $\lambda_1 - 1 > 0$ and $l - \lambda_1 - 1 > 0$, then the statement is clear. Let us suppose otherwise. First, let $\lambda_1 - 1 = 0$. If $l - \lambda_1 - 1 = 0$, then

$$k_i = \iota^{\lambda_1}(0) \mathbin{+\!\!+} (\lambda_2) \mathbin{+\!\!+} (\lambda_2) \mathbin{+\!\!+} (\lambda_2).$$

Claim 1 follows, since $L(l,\lambda_1) = H(l,\lambda_1) = 0$. If $l - \lambda_1 - 1 > 0$, then

$$k_i = \iota^{\lambda_1}(0) \mathbin{+\!\!+} (\lambda_2) \mathbin{+\!\!+} \iota^{l-\lambda_1-1}(0) \mathbin{+\!\!+} (\lambda_2) \mathbin{+\!\!+} (\lambda_2) \mathbin{+\!\!+} \iota^{l-\lambda_1-1}(0).$$

Claim 1 holds by $L(l,\lambda_1) = 0 < H(l,\lambda_1) = l - \lambda_1 - 1$.

Second, let $\lambda_1 - 1 > 0$ and $l - \lambda_1 - 1 = 0$. Then,

$$k_i = \iota^{\lambda_1}(0) \mathbin{+\!\!+} (\lambda_2) \mathbin{+\!\!+} (\lambda_2) \mathbin{+\!\!+} \iota^{\lambda_1-1}(0) \mathbin{+\!\!+} (\lambda_2).$$

This gives Claim 1 by $L(l,\lambda_1) = 0 < H(l,\lambda_1) = \lambda_1 - 1$.

Proof of Claim 2. We prove that each $P_{s,l,\lambda,i} = (s,l,m_i,n_i,k_i)$ is fitting. Since s is an infinite gate, $m_i, n_i \in [\![s]\!]^l$. Then, we obtain equation (12.5.5), ignoring $\mathbin{+\!\!+}\iota^{\lambda_1-1}(0)$ or $\mathbin{+\!\!+}\iota^{l-\lambda_1-1}(0)$ in k_i when $\lambda_1 - 1 = 0$ or $l - \lambda_1 - 1 = 0$. Also, we have

$$\nu(m_i,n_i,k_i) = \iota^l(\lambda_2 + \lambda_3 + s_1) \mathbin{+\!\!+} \iota^l(3\lambda_2 + \lambda_3 + s_1) + (i-1)\iota^{2l}(2\lambda_2). \tag{12.5.6}$$

By equations (12.5.5) and (12.5.6), condition (12.5.4) gives the slope conditions on $\nu(k_i)$ and $\nu(m_i,n_i,k_i)$. It follows that each $P_{s,l,\lambda,i}$ is fitting.

For each $i \in [\![\theta]\!]$, we prove that $Q_i = \left(Q_{i,\kappa} = P_{s,l,\lambda,i+\kappa}\right)_{\kappa \in [\![0,1]\!]}$ is a fitting path of type e-e. First, assume $L(l,\lambda_1) = l - \lambda_1 - 1$. Since $\lambda_1 - 1 - (l - \lambda_1 - 1) = 2\lambda_1 - l \geq 0$, Claim 1 gives

$$\sigma(k_i) = \iota^{l-\lambda_1}(3\lambda_2) \mathbin{+\!\!\!+} \iota^{2\lambda_1 - l}(2\lambda_2) \mathbin{+\!\!\!+} \iota^{l-\lambda_1}(\lambda_2).$$

Then, $l - \lambda_1 + 2\lambda_1 - l = \lambda_1$ implies

$$n_i \boxplus k_i = \iota^{l-\lambda_1}(2\lambda_2) \mathbin{+\!\!\!+} \iota^{\lambda_1}(3\lambda_2) + \iota^{l}(\lambda_3 + s_1) + (i-1)\iota^{l}(2\lambda_2).$$

By $(n_i \boxplus k_i)^{\vee} = m_{i+1}$, we deduce that each Q_i is a fitting path of type e-e.

Second, assume $L(l,\lambda_1) = \lambda_1 - 1$. Since $l - \lambda_1 - 1 - (\lambda_1 - 1) = l - 2\lambda_1 \geq 0$, Claim 1 yields

$$\sigma(k_i) = \iota^{\lambda_1}(3\lambda_2) \mathbin{+\!\!\!+} \iota^{l-2\lambda_1}(2\lambda_2) \mathbin{+\!\!\!+} \iota^{\lambda_1}(\lambda_2).$$

Furthermore, $\lambda_1 + l - 2\lambda_1 = l - \lambda_1$ gives

$$n_i \boxplus k_i = \iota^{l-\lambda_1}(2\lambda_2) \mathbin{+\!\!\!+} \iota^{\lambda_1}(3\lambda_2) + \iota^{l}(\lambda_3 + s_1) + (i-1)\iota^{l}(2\lambda_2).$$

Since $(n_i \boxplus k_i)^{\vee} = m_{i+1}$, each Q_i is a fitting path of type e-e. □

Example 12.23. Let $s = (0,\infty)$, $l = 3$, and $\lambda = (1,1,0)$. Then, $\theta = (1,\infty)$ gives the infinite-length fitting path $P_{s,l,\lambda} = \left(P_{s,l,\lambda,i} = (s,l,m_i,n_i,k_i)\right)_{i \in [\![\theta]\!]}$ of type e-e such that each $i \in [\![\theta]\!]$ satisfies

$$\begin{aligned} m_i &= n_i = (1,0,0) + 2i - 2, \\ k_i &= (0,1,0,1,1,0). \end{aligned}$$

12.6 Triplet scalings and sums of fitting paths

To construct more fitting paths, we introduce the following triplet scalings and sums.

Definition 12.24. *Suppose a gate s and $l \in \mathbb{Z}_{\geq 1}$.*

1. *Assume a tuple $T = (s,l,\alpha,\beta,\gamma)$ such that $\alpha,\beta \in \mathbb{Z}^l$ and $\gamma \in \mathbb{Z}^{2l}$. Let $\lambda \in \mathbb{Z}$. Then, we define the triplet scaling*

$$\lambda \odot T = (s,l,\lambda\alpha,\lambda\beta,\lambda\gamma).$$

2. *For each $i \in [\![2]\!]$, assume a tuple $P_i = (s,l,m_i,n_i,k_i)$ such that $m_i,n_i \in \mathbb{Z}^l$ and $k_i \in \mathbb{Z}^{2l}$. Then, we define the triplet sum*

$$P_1 \dotplus P_2 = (s,l,m_1+m_2,n_1+n_2,k_1+k_2).$$

In particular, we obtain the following fitting tuples.

Lemma 12.25. *Consider fitting tuples $P_j = (s,l,m_j,n_j,k_j)$ for $j \in [\![2]\!]$. Suppose $\lambda \in \mathbb{Z}^2$ such that*

$$\lambda_1, \lambda_2 \geq 0 \text{ and } \lambda_1 + \lambda_2 > 0. \tag{12.6.1}$$

Let $P_3 = \lambda_1 \odot P_1 \dotplus \lambda_2 \odot P_2 = (s,l,m_3,n_3,k_3)$. Then, P_3 is fitting, provided $m_3,n_3 \in [\![s]\!]^l$.

Proof. Let $a_i = \nu(k_i)$ and $b_i = \nu(m_i, n_i, k_i)$ for $i \in [\![3]\!]$. Then, we have

$$\begin{aligned} a_3 &= \lambda_1 \nu(k_1) + \lambda_2 \nu(k_2) \\ &= \lambda_1 a_1 + \lambda_2 a_2, \\ b_3 &= a_3 + m_3 \mathbin{+\!\!\!+} n_3 \\ &= a_3 + (\lambda_1 m_1 + \lambda_2 m_2) \mathbin{+\!\!\!+} (\lambda_1 n_1 + \lambda_2 n_2) \\ &= \lambda_1 a_1 + \lambda_2 a_2 + \lambda_1 m_1 \mathbin{+\!\!\!+} \lambda_1 n_1 + \lambda_2 m_2 \mathbin{+\!\!\!+} \lambda_2 n_2 \\ &= \lambda_1 b_1 + \lambda_2 b_2. \end{aligned}$$

Thus, the slope conditions of P_1 and P_2 imply those of P_3 by inequalities (12.6.1). This gives the statement by $m_3, n_3 \in [\![s]\!]^l$. □

We now introduce the following triplet scalings and sums on tuple sequences.

Definition 12.26. *Suppose gates s and θ. Let $l \in \mathbb{Z}_{\geq 1}$.*

1. *Consider a sequence $T = (T_i = (s, l, \alpha_i, \beta_i, \gamma_i))_{i \in [\![\theta]\!]}$ with $\alpha_i, \beta_i \in \mathbb{Z}^l$ and $\gamma_i \in \mathbb{Z}^{2l}$. Let $\lambda \in \mathbb{Z}$. Then, we define the triplet scaling*

$$\lambda \odot T = (\lambda \odot T_i)_{i \in [\![\theta]\!]}.$$

2. *For each $j \in [\![2]\!]$, consider a sequence $P_j = (P_{j,i} = (s, l, m_{j,i}, n_{j,i}, k_{j,i}))_{i \in [\![\theta]\!]}$ with $m_{j,i}, n_{j,i} \in \mathbb{Z}^l$ and $k_{j,i} \in \mathbb{Z}^{2l}$. Then, we define the triplet sum*

$$P_1 \dotplus P_2 = (P_{1,i} \dotplus P_{2,i})_{i \in [\![\theta]\!]}.$$

For example, suppose an infinite gate $s \geq 0$ and $\lambda = (\lambda_1, \lambda_2, 0) \in \mathbb{Z}^3$ such that $s_1 = 0$ and $\lambda_1, \lambda_2 \in \mathbb{Z}_{\geq 1}$. This gives $P_{s,l,\lambda} = \lambda_2 \odot P_{s,l,(\lambda_1,1,0)}$.

We now prove the following strict inequality on the ladders of fitting paths.

Lemma 12.27. *Assume a fitting path $P = ((s, l, m_i, n_i, k_i))_{i \in [\![1,2]\!]}$. Then, we have*

$$n_1 = m_1 < m_2^\vee \text{ or } n_1 = m_1^\vee < m_2^\vee.$$

In particular, we have

$$\sum m_1 = \sum n_1 < \sum m_2 = \sum n_2.$$

Proof. Since P is a fitting path, $n_1 = m_1$ or $n_1 = m_1^\vee$. If $n_1 = m_1$, then $\sigma(k) > 0$ gives

$$m_1 < m_1 \boxplus k_1 = n_1 \boxplus k_1 = m_2^\vee.$$

If $n_1 = m_1^\vee$, then

$$m_1^\vee < m_1^\vee + \sigma(k_1)^\vee = n_1 + \sigma(k_1)^\vee = n_1 \boxplus k_1 = m_2^\vee.$$

The latter statement follows from $m_2 = n_2$ or $m_2^\vee = n_2$. □

Then, we obtain fitting paths by triplet scalings and sums.

Proposition 12.28. *Assume $\lambda \in \mathbb{Z}^2_{\geq 1}$ such that*

$$\lambda_1, \lambda_2 \geq 0 \text{ and } \lambda_1 + \lambda_2 > 0. \tag{12.6.2}$$

Let $\theta = (1,2)$. Consider fitting paths $P_j = (P_{j,i} = (s, l, m_{j,i}, n_{j,i}, k_{j,i}))_{i \in [\![\theta]\!]}$ for $j \in [\![2]\!]$ such that P_1 and P_2 have the same type A. Also, suppose $m_{3,2}, n_{3,2} \in [\![s]\!]^l$ and

$$P_3 = \lambda_1 \odot P_1 \dotplus \lambda_2 \odot P_2 = (P_{3,i} = (s, l, m_{3,i}, n_{3,i}, k_{3,i}))_{i \in [\![\theta]\!]}.$$

Then, P_3 is a fitting path of the type A.

Proof. Lemma 12.25 implies that $P_{3,2}$ is fitting. Hence, let us prove that $P_{3,1}$ is fitting. Thus, we confirm $m_{3,1}, n_{3,1} \in [\![s]\!]^l$. By inequalities (12.6.2), we have

$$m_{3,1} \geq m_{1,1} \text{ or } m_{2,1}, \tag{12.6.3}$$
$$n_{3,1} \geq n_{1,1} \text{ or } n_{2,1}. \tag{12.6.4}$$

Also, since P_1 and P_2 are fitting paths, Lemma 12.27 yields

$$m_{3,1} = \lambda_1 m_{1,1} + \lambda_2 m_{2,1} < \begin{cases} \lambda_1 m_{1,2} + \lambda_2 m_{2,2} = m_{3,2} \text{ if A is type f-e or f-f,} \\ \lambda_1 m_{1,2}^\vee + \lambda_2 m_{2,2}^\vee = m_{3,2}^\vee \text{ if A is type e-e or e-f.} \end{cases}$$

Thus, $m_{3,1} \in [\![s]\!]^l$ by $m_{3,2} \in [\![s]\!]^l$ and inequality (12.6.3). Also, $n_{3,1} \in [\![s]\!]^l$ by $m_{3,2} \in [\![s]\!]^l$ and inequality (12.6.4), because Lemma 12.27 gives

$$n_{3,1} = \lambda_1 n_{1,1} + \lambda_2 n_{2,1} < \lambda_1 m_{1,2}^\vee + \lambda_2 m_{2,2}^\vee = m_{3,2}^\vee.$$

Hence, $P_{3,1}$ is fitting by Lemma 12.25.

Let us prove that P_3 is a fitting path of the type A. We have

$$m_{3,2} = (n_{3,1} \boxplus k_{3,1})^\vee, \tag{12.6.5}$$

since

$$\begin{aligned} n_{3,1} \boxplus k_{3,1} &= (\lambda_1 n_{1,1} + \lambda_2 n_{2,1}) \boxplus (\lambda_1 k_{1,1} + \lambda_2 k_{2,1}) \\ &= \lambda_1 (n_{1,1} \boxplus k_{1,1}) + \lambda_2 (n_{2,1} \boxplus k_{2,1}) \\ &= \lambda_1 m_{1,2}^\vee + \lambda_2 m_{2,2}^\vee \\ &= m_{3,2}^\vee. \end{aligned}$$

For instance, assume that P_1 and P_2 have the type e-e. Then, we have $m_{3,1} = n_{3,1}$ by $m_{1,1} = n_{1,1}$ and $m_{2,1} = n_{2,1}$, and $m_{3,2} = n_{3,2}$ by $m_{1,2} = n_{1,2}$ and $m_{2,2} = n_{2,2}$. Thus, P_3 is of type e-e by equation (12.6.5). Similar discussions hold for the other types. □

12.7 Infinite-length fitting paths of Young diagrams

By box counting, we prove that Young diagrams give explicit infinite-length fitting paths. We adopt the following notation.

Definition 12.29. *Suppose $p, \lambda \in \mathbb{Z}_{\geq 1}$.*

1. *$\xi \in \mathbb{Z}_{\geq 1}^{p}$ is called a Young diagram (or a partition of $\sum \xi$) if ξ is decreasing.*
2. *We write $Y(p,\lambda)$ for the set of Young diagrams $\xi \in \mathbb{Z}_{\geq 1}^{p}$ such that $\xi_1 \leq \lambda$.*

By Definition 2.1, we use the following notation for the box counting of $Y(p,\lambda)$.

Definition 12.30. *Let $\xi \in Y(p,l)$, $\lambda \in \mathbb{Z}$, and $\lambda' \in \mathbb{Z}_{\geq 1}$.*

1. *Let $\mathrm{L}_{\leq \lambda}(\xi) = \sum_{i \leq \lambda} \mathrm{L}_i(\xi) \in \mathbb{Z}_{\geq 0}$.*
2. *Let $\mathrm{L}(\lambda,\xi) = 2\mathrm{L}_{\leq \lambda-1}(\xi) + \mathrm{L}_{\lambda}(\xi) \in \mathbb{Z}_{\geq 0}$.*
3. *Let $z(\lambda',\lambda,\xi) = \lambda + (\mathrm{L}_{\leq 0}(\xi), \mathrm{L}_{\leq 1}(\xi), \ldots, \mathrm{L}_{\leq \lambda'-1}(\xi)) \in \mathbb{Z}^{\lambda'}$.*

Moreover, we define the following sequence of fitting tuples.

Definition 12.31. *Let $h \in \mathbb{Z}_{\geq 0}$ and $\xi \in Y(p,l)$. Consider $\lambda_i \in \mathbb{Z}_{\geq 0}^{3}$ for $i \in [\![p]\!]$ such that $\lambda_1 = (\xi_1, 1, h)$ and $\lambda_i = (\xi_i, 1, 0)$ for $i \in [\![2,p]\!]$. Also, suppose an infinite gate $s \geq 0$. Let $l \in \mathbb{Z}_{\geq 1}$, $\theta = (1,\infty)$, and $t = (t_i \in \mathbb{Z}_{\geq 0})_{i \in [\![\theta]\!]}$. Then, we define the sequence*

$$\mathbb{P}_{s,l,\xi,t,h} = P_{s,l,\lambda_1,t} \dotplus P_{s,l,\lambda_2} \dotplus \cdots \dotplus P_{s,l,\lambda_p}.$$

We call h the base height of $\mathbb{P}_{s,l,\xi,t,h}$. If each $t_i = 0$, then we also write $\mathbb{P}_{s,l,\xi,h}$ for $\mathbb{P}_{s,l,\xi,t,h}$.

Then, we realize the following infinite-length fitting paths $\mathbb{P}_{s,l,\xi,t,h}$.

Theorem 12.32. *Consider a Young diagram $\xi \in Y(p,l)$. Then, $\mathbb{P}_{s,l,\xi,t,h} = ((s,l,m_i,n_i,k_i))_{i \in [\![\theta]\!]}$ is an infinite-length fitting path of type e-e. Moreover, each $i \in [\![\theta]\!]$ satisfies*

$$m_i = n_i = p + s_1 + h - z(l,0,\xi) + (i-1)\iota^l(\mathrm{L}(l,\xi)), \tag{12.7.1}$$
$$\nu(k_i) = z(l,0,\xi) \mathbin{+\!\!+} z(l,\mathrm{L}(l,\xi),\xi) + t_i. \tag{12.7.2}$$

Proof. The former statement holds by Propositions 12.20, 12.22, and 12.28.

We prove equations (12.7.1) and (12.7.2). Suppose $s_1 = h = 0$ and each $t_i = 0$ for simplicity. Thus, let $\lambda_j = (\xi_j, 1, 0)$ for $j \in [\![p]\!]$. Also, consider the fitting path $P_{s,l,\lambda_j} = \left(\left(s,l,m_{\lambda_j,i},n_{\lambda_j,i},k_{\lambda_j,i}\right)\right)_{i \in [\![\theta]\!]}$ for $j \in [\![p]\!]$. Then, Propositions 12.20 and 12.22 give the following equations:

$$m_{\lambda_j,i} = n_{\lambda_j,i} = \begin{cases} \iota^l(1) + (i-1)\iota^l(1) \text{ if } \xi_j = l, \\ \iota^{\xi_j}(1) \mathbin{+\!\!+} \iota^{l-\xi_j}(0) + (i-1)\iota^l(2) \text{ if } \xi_j < l; \end{cases} \tag{12.7.3}$$
$$\nu(k_{\lambda_j,i}) = \begin{cases} \iota^l(0) \mathbin{+\!\!+} \iota^l(1) \text{ if } \xi_j = l, \\ \iota^{\xi_j}(0) \mathbin{+\!\!+} \iota^{l-\xi_j}(1) \mathbin{+\!\!+} \iota^{\xi_j}(2) \mathbin{+\!\!+} \iota^{l-\xi_j}(3) \text{ if } \xi_j < l. \end{cases} \tag{12.7.4}$$

Thus, equation (12.7.1) holds by equation (12.7.3), since each $\kappa \in [\![l]\!]$ satisfies

$$m_{1,\kappa} = n_{1,\kappa} = \sum_{j \in [\![p]\!]} m_{\lambda_j,1,\kappa} = \sum_{u \in [\![\kappa,l]\!]} \mathrm{L}_u(\xi).$$

Also, equation (12.7.2) holds, since equation (12.7.4) gives

$$\nu(k_i) = \iota^l(0) + \iota^l(\mathrm{L}_l(\xi)) + \sum_{\kappa \in [\![l-1]\!]} \iota^\kappa(0) + \iota^{l-\kappa}(\mathrm{L}_\kappa(\xi)) + \iota^\kappa(2\mathrm{L}_\kappa(\xi)) + \iota^{l-\kappa}(3\mathrm{L}_\kappa(\xi)).$$

□

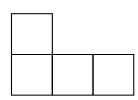
Theorem 12.32 gives not only infinitely many polynomials with positive integer coefficients by Young diagrams and the merged-log-concavity, but also infinite-length almost strictly unimodal sequences by Theorem 12.18. Also, $z(l,0,\xi)$ and $p - z(l,0,\xi)$ in Theorem 12.32 count boxes of Young diagrams in $p \times l$ rectangles as in the following examples.

Example 12.33. Let $l = 4$, $p = 2$, $\xi = (3,1)$, $s = (0,\infty)$, and $h = 0$. Then, there exists the infinite-length fitting path $\mathbb{P}_{s,l,\xi,h} = ((s,l,m_i,n_i,k_i))_{i \in [\![\theta]\!]}$ such that each $i \in [\![\theta]\!]$ satisfies

$$m_i = n_i = (2,1,1,0) + (i-1)\iota^l(4), \tag{12.7.5}$$

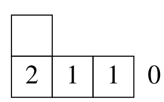
$$\nu(k_i) = (0,1,1,2,4,5,5,6). \tag{12.7.6}$$

To explain these equations by $p - z(l,0,\xi)$ and $z(l,0,\xi)$, consider the following Young diagram of ξ:



.

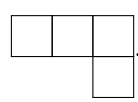
Then, $p - z(l,0,\xi)$ corresponds to the following box counting:



,

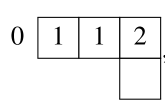
where $2,1,1,0$ indicate the numbers of boxes in the vertical direction. Furthermore, equation (12.7.5) follows from Theorem 12.32, since $\mathrm{L}_{\leq l-1}(\xi) = 2$ and $\mathrm{L}_l(\xi) = 0$ gives $\mathrm{L}(l,\xi) = 4$.

Moreover, in the $p \times l$ rectangle, ξ gives the following complementary Young diagram:



.

Then, $z(l,0,\xi)$ corresponds to the following box counting:



,

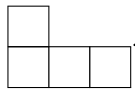
where $0,1,1,2$ are the numbers of boxes in the vertical direction. Thus, equation (12.7.6) follows.

When $\xi_1 = l$, we have the following example.

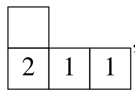
Example 12.34. Let $l=3$, $p=2$, $\xi=(3,1)$, $s=(0,\infty)$, and $h=0$. Then, there is the infinite-length fitting path $\mathbb{P}_{s,l,\xi,h}=((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$ such that each $i\in[\![\theta]\!]$ satisfies

$$m_i=n_i=(2,1,1)+(i-1)\iota^l(3), \tag{12.7.7}$$
$$\nu(k_i)=(0,1,1,3,4,4)\,. \tag{12.7.8}$$

To explain these equations, we have the following Young diagram of ξ:

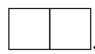


Thus, $p-z(l,0,\xi)$ corresponds to the following box counting:

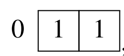


where $2,1,1$ are the numbers of boxes in the vertical direction. This gives equation (12.7.7) by Theorem 12.32, since $\mathrm{L}_{\leq l-1}(\xi)=1$ and $\mathrm{L}_l(\xi)=1$.

Furthermore, in the $p\times l$ rectangle, ξ has the following complementary Young diagram:



Then, $z(l,0,\xi)$ corresponds to the following box counting:



where $0,1,1$ are the numbers of boxes in the vertical direction. Hence, we obtain equation (12.7.8).

Remark 12.35. Let $l\in\mathbb{Z}_{\geq 1}$, $p=1$, $\xi=(l)$, $s=(0,\infty)$, and $h=0$. Then, there is the infinite-length fitting path $\mathbb{P}_{s,l,\xi,h}=((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$ such that each $i\in[\![\theta]\!]$ satisfies

$$m_i=n_i=i\iota^l(1),$$
$$\nu(k_i)=\iota^l(0)+\iota^l(1).$$

Furthermore, suppose a parcel $\mathscr{F}=\Lambda(s,l,\succ,f_s,\mathfrak{X})$ and $i\in[\![\theta]\!]$. Then, since $\sigma(k)=\iota^l(1)$, the $\succ'$-multi-log-concavity

$$\mathscr{F}_{m_i}\mathscr{F}_{m_i}-\mathscr{F}_{m_i-1}\mathscr{F}_{m_i+1}\succ' 0$$

coincides with the $\succ'$-merged-log-concavity

$$\mathscr{F}_{m_i}\mathscr{F}_{n_i^\vee}-\mathscr{F}_{m_i\boxminus k_i}\mathscr{F}_{(n_i\boxplus k_i)^\vee}\succ' 0.$$

We now obtain infinite-length almost strictly unimodal sequences by the following lemma.

Lemma 12.36. *Suppose $P = \mathbb{P}_{s,l,\xi,t,h} = ((s,l,m_i,n_i,k_i))_{i \in [\![\theta]\!]}$. Then, P is wrapped when*

$$h \geq 3\mathrm{L}_{\leq l-1}(\xi). \tag{12.7.9}$$

Proof. Since s is infinite, P has the upper inclusion condition. Let us confirm that inequality (12.7.9) gives the lower inclusion condition $m_1 \boxminus k_1 \geq s_1$ of P. Let $a_1 = \nu(k_1)$ and $p = \mathrm{L}_{\leq l-1}(\xi) + \mathrm{L}_l(\xi)$. Then, Theorem 12.32 gives

$$\sigma(k_1)_1 = a_{1,2l} - t_1 = \mathrm{L}(l,\xi) + \mathrm{L}_{\leq l-1}(\xi) = 3\mathrm{L}_{\leq l-1}(\xi) + \mathrm{L}_l(\xi) = 2\mathrm{L}_{\leq l-1}(\xi) + p.$$

Thus, since $m_1 \geq s_1 + h + p - \mathrm{L}_{\leq l-1}(\xi)$ by Theorem 12.32, we obtain

$$m_1 \boxminus k_1 \geq s_1 + h + p - \mathrm{L}_{\leq l-1}(\xi) - \sigma(k_1)_1 = s_1 + h - 3\mathrm{L}_{\leq l-1}(\xi).$$

In particular, inequality (12.7.9) gives the lower inclusion condition of P. □

Proposition 12.37. *Let $P = \mathbb{P}_{s,l,\xi,h} = ((s,l,m_i,n_i,k_i))_{i \in [\![\theta]\!]}$. Suppose that $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ is $\succ$-merged-log-concave and flip-invariant at some $r \in \mathrm{O}_{\mathfrak{X}}$. Let $\zeta = (P,\mathscr{F})$. Then, we have the following.*

1. *$u(\zeta,r)$ is infinite-length and strictly log-concave.*
2. *$u(\zeta,r)$ or $-u(\zeta,r)$ is infinite-length and almost strictly unimodal if $h \geq 3\mathrm{L}_{\leq l-1}(\xi)$.*

Proof. *Proof of Claim 1.* Claim 1 is of Theorems 12.18 and 12.32.
Proof of Claim 2. Lemma 12.36 and Claim 1 imply Claim 2. □

13 Almost strictly unimodal sequences and critical points

We have the notions of increasing, decreasing, and hill sequences of real numbers. Also, we adopt the following notion of infinite-length sequences, extending the notion of hill sequences to asymptotic cases.

Definition 13.1. *Consider a strictly increasing sequence $r = (r_i \in \mathbb{R})_{i \in [\![\theta]\!]}$ for an infinite gate θ. Then, r is an asymptotic hill if*

$$\lim_{i \to \infty} \frac{r_{i+1}}{r_i} = 1.$$

Using the notions of almost strictly unimodal sequences, we have the following boundary sequences, which satisfy two of the notions simultaneously:

- hill and increasing sequences;
- hill and decreasing sequences;
- asymptotically hill and increasing sequences.

By these boundary sequences, we consider critical points on the variation of almost strictly unimodal sequences. We also obtain real algebraic sets of the critical points in a suitable setting.

13.1 Increasing, hill, and decreasing sequences

Since hill sequences have at least three terms in Definition 1.1, we state the following classification of almost strictly unimodal sequences.

Lemma 13.2. *Let $r = (r_i \in \mathbb{R})_{i \in [\![s]\!]}$ be almost strictly unimodal with $s_2 - s_1 \geq 2$. Let δ be the mode of r.*

1. *If $\delta = s_2$, then r is strictly increasing.*
2. *If $\delta = s_1$ and $r_\delta > r_{\delta+1}$, then r is strictly decreasing.*
3. *In other cases, r is a hill.*

Proof. Proof of Claim 1. If $s_2 < \infty$, then $r = (r_i)_{i \leq \delta}$ is strictly increasing. If $s_2 = \infty$, then r_δ and $r_{\delta+1}$ are non-existent.

Proof of Claim 2. Claim 2 holds, since $(r_i)_{i>\delta}$ is strictly decreasing.

Proof of Claim 3. If $s_1 < \delta < s_2$, then $\delta < \infty$ gives hill sequences. If $\delta = s_1$ and $r_\delta = r_{\delta+1}$, then $s_2 - s_1 \geq 2$ gives hill sequences. □

We establish the following criteria for some boundary sequences.

Proposition 13.3. *Consider an almost strictly unimodal $r = (r_i \in \mathbb{R})_{i \in [\![s]\!]}$ such that $s_2 - s_1 \geq 2$. Then, we have the following.*

1. *r is a hill and decreasing sequence if and only if $r_{s_1} = r_{s_1+1}$.*
2. *r is a hill and increasing sequence if and only if r satisfies the following conditions:*

$$s_2 < \infty; \tag{13.1.1}$$

$$r_{s_2-1} = r_{s_2}. \tag{13.1.2}$$

Proof. Assume the mode δ of r.

Proof of Claim 1. First, the if part follows, because $s_2 - s_1 \geq 2$ gives $r_{s_1} = r_{s_1+1} > r_{s_1+2} > \dots$, which is a hill and decreasing sequence. Second, the only if part holds as follows. If $s_1 < \delta \leq s_2$, then r is not decreasing by $r_{s_1} < r_{s_1+1}$. If $s_1 = \delta$, then $r_\delta = r_{\delta+1}$ by Claim 2 of Lemma 13.2.

Proof of Claim 2. First, the if part holds, since $s_2 - s_1 \geq 2$ gives $\cdots < r_{s_2-2} < r_{s_2-1} = r_{s_2}$, which is increasing and hill. Second, let us prove the only if part. If $s_2 = \infty$, then r is not increasing when $\delta < \infty$, or strictly increasing when $\delta = \infty$ by Claim 1. Hence, inequality 13.1.1 holds. Also, the strictly decreasing part $(r_i)_{i>\delta}$ has to be trivial. It follows that $\delta = s_2 - 1$ by Claim 1 of Lemma 13.2. We obtain equation (13.1.2) for the increasing r. □

13.2 Critical points on almost strictly unimodal sequences

We first introduce the notion of merged pairs.

Definition 13.4. *Assume a wrapped fitting path $P = ((s, l, m_i, n_i, k_i))_{i \in [\![\theta]\!]}$ and a parcel $\mathscr{F} = \Lambda(s, l, w, \succ, f_s, \phi, \rho, x, \mathfrak{X})$ that is flip-invariant and $>_{\mathrm{O}_{\mathfrak{X}}}$-merged-log-concave. Then, we call the tuple $(P, \mathscr{F})$ a $(\theta, \mathfrak{X})$-merged pair.*

Suppose a $(\theta, \mathfrak{X})$-merged pair $\xi = (P, \mathscr{F})$ and $r \in \mathrm{O}_{\mathfrak{X}}$. Then, we define the following critical points on $u(\xi, r)$, which is an almost strictly unimodal sequence by Theorem 12.18.

Definition 13.5. *Suppose a $(\theta,\mathfrak{X})$-merged pair $\zeta=(P,\mathscr{F})$. Let $r\in \mathrm{O}_{\mathfrak{X}}$.*

1. *We call r a front critical point of ζ if $u(\zeta,r)$ is a hill and decreasing sequence.*
2. *We call r a rear critical point of ζ if $u(\zeta,r)$ is a hill and increasing sequence.*

We employ the term "front critical point", since $u(\zeta,r)$ is a hill and decreasing sequence when its front terms satisfy $u(\zeta,r)_{\theta_1-1}=u(\zeta,r)_{\theta_1}$ by Proposition 13.3. Similarly, we employ the term "rear critical point", since $u(\zeta,r)$ of $\theta_2<\infty$ is a hill and increasing sequence when its rear terms satisfy $u(\zeta,r)_{\theta_2}=u(\zeta,r)_{\theta_2+1}$.

Rear critical points are only for finite-length fitting paths, since we do not have hill and increasing infinite-length sequences among almost strictly unimodal sequences. But, suppose an infinite gate θ. Then, there is a $(\theta,\mathfrak{X})$-merged pair $\lambda=(P,\mathscr{G})$ with $b_1,b_2,b_3\in \mathrm{O}_{\mathfrak{X}}$ such that

- $u(\lambda,r)$ is a hill if $b_1\le r<b_2$, and
- $u(\lambda,r)$ is a strictly increasing sequence if $b_2\le r\le b_3$.

We compute such a merged pair in some detail in Section 14.2.

For infinite-length merged pairs, let us introduce the following notion of asymptotic critical points, which is analogous to the notion of rear critical points.

Definition 13.6. *Suppose a $(\theta,\mathfrak{X})$-merged pair $\zeta=(P,\mathscr{F})$ of an infinite gate θ. We call $r\in \mathrm{O}_{\mathfrak{X}}$ an asymptotic critical point of ζ if*

$$\lim_{j\to\infty}\frac{u(\zeta,r)_{j+1}}{u(\zeta,r)_j}=1.$$

By the asymptotic critical points, we obtain asymptotic hills.

Proposition 13.7. *For a $(\theta,\mathfrak{X})$-merged pair $\zeta=(P,\mathscr{F})$, assume an asymptotic critical point $r\in \mathrm{O}_{\mathfrak{X}}$. Then, $u(\zeta,r)$ is an asymptotic hill.*

Proof. We prove that $u(\zeta,r)$ is strictly increasing. Assume that $u(\zeta,r)$ is a hill or decreasing sequence. This gives $\lambda\in[\![\theta]\!]$ such that $u(\zeta,r)_\lambda>u(\zeta,r)_{\lambda+1}>\dots$. Since $u(\zeta,r)>0$ is strictly log-concave by Theorem 12.18, we have $1\ge\frac{u(\zeta,r)_{\lambda+1}}{u(\zeta,r)_\lambda}>\frac{u(\zeta,r)_{\lambda+2}}{u(\zeta,r)_{\lambda+1}}>\dots$ against r being an asymptotic critical point. □

13.3 Parcel ratios and vanishing constraints of parcel numerators

Suppose a merged pair $(P,\mathscr{F})$ with an infinite-length $P=((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$. Then, we discuss parcel ratios $\frac{\mathscr{F}_{n_i\boxplus k_i}}{\mathscr{F}_{n_i}}$ by almost strictly unimodal sequences and polynomials. This gives certain vanishing constraints on parcel numerators.

We state the lemma below on infinite-length fitting paths.

Lemma 13.8. *Assume an infinite-length fitting path $P=((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$. Then, $\lim_{i\to\infty}n_{i,j}=\infty$ for each $j\in[\![l]\!]$. In particular, s is an infinite gate.*

Proof. Let $i\in[\![\theta]\!]$ such that $i\ge\theta_1+1$. Then, n_i is $n_{i-1}\boxplus k_{i-1}$ or $(n_{i-1}\boxplus k_{i-1})^\vee$. Thus, $n_i\ge i-\theta_1\in\mathbb{Z}_{\ge1}$ by $\sigma(k)\ge1$ and $n_{\theta_1}\ge0$. So, we have the former assertion. This also gives the latter assertion by the inclusion condition of each fitting tuple of P. □

We also have the following lemma on parcel ratios.

Lemma 13.9. *Consider a merged pair $(P,\mathscr{F})$ such that $P = ((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$ and $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$. Let ψ be the l-canonical mediator and $\kappa = k_{\theta_1}$. For each $j \in [\![\theta]\!]$, let*

$$G(j,x) = \prod_{i\in[\![l]\!],\lambda\in[\![\sigma(\kappa)^\vee_i]\!]} (1-x_i^{n_{j,i}+\lambda})^{w_i}.$$

Then, we have the following.

1. *If $j \in [\![\theta]\!]$, then*

$$\frac{\mathscr{F}_{n_j\boxplus k_j}}{\mathscr{F}_{n_j}} = \frac{f_{s,n_j\boxplus\kappa}}{f_{s,n_j}} \cdot \frac{\prod \psi(x)^{\sigma(\kappa)^\vee\circ w}}{\prod \phi(x)^{\sigma(\kappa)^\vee\circ w}} \cdot \frac{1}{G(j,x)}.$$

2. *If P is infinite-length, then $\lim_{j\to\infty} G(j,x)|_{x=(x_1(r),\dots,x_l(r))} = 1$ for each $r \in \mathrm{O}_{\mathfrak{X}}$.*

Proof. Proof of Claim 1. For equivalent supports k_j and κ, $\sigma(k_j) = \sigma(\kappa)$ and $n_j \boxplus k_j = n_j \boxplus \kappa$. Then, since $n_j \boxplus k_j - n_j = \sigma(\kappa)^\vee$, we obtain

$$\begin{aligned}
\frac{\mathscr{F}_{n_j\boxplus k_j}}{\mathscr{F}_{n_j}} &= \frac{\prod \phi(x)^{n_j\circ w}\cdot [n_j]!^w_x}{f_{s,n_j}} \cdot \frac{f_{s,n_j\boxplus k_j}}{\prod \phi(x)^{(n_j\boxplus k_j)\circ w}\cdot [n_j\boxplus k_j]!^w_x} \\
&= \frac{f_{s,n_j\boxplus\kappa}}{f_{s,n_j}} \cdot \frac{1}{\prod \phi(x)^{\sigma(\kappa)^\vee\circ w}} \cdot \frac{[n_j]!^w_x}{[n_j\boxplus\kappa]!^w_x} \\
&= \frac{f_{s,n_j\boxplus\kappa}}{f_{s,n_j}} \cdot \frac{\prod \psi(x)^{\sigma(\kappa)^\vee\circ w}}{\prod \phi(x)^{\sigma(\kappa)^\vee\circ w}} \cdot \frac{[n_j]!^w_x}{\prod_{i\in[\![l]\!]}(1-x_i)^{\sigma(\kappa)^\vee_i\cdot w_i}\cdot [n_j\boxplus\kappa]!^w_x}.
\end{aligned}$$

Thus, Claim 1 holds, since

$$\begin{aligned}
\prod_{i\in[\![l]\!]}(1-x_i)^{\sigma(\kappa)^\vee_i\cdot w_i}\cdot\frac{[n_j\boxplus\kappa]!^w_x}{[n_j]!^w_x} &= \prod_{i\in[\![l]\!]}(1-x_i)^{\sigma(\kappa)^\vee_i\cdot w_i}\cdot[n_{j,i}+1]^{w_i}_{x_i}\dots[n_{j,i}+\sigma(\kappa)^\vee_i]^{w_i}_{x_i} \\
&= \prod_{i\in[\![l]\!]}(1-x_i^{n_{j,i}+1})^{w_i}\dots(1-x_i^{n_{j,i}+\sigma(\kappa)^\vee_i})^{w_i}.
\end{aligned}$$

Proof of Claim 2. Claim 2 holds by Lemma 13.8, since $0 < x_i(r) < 1$ by Claim 1 of Lemma 5.22. □

We introduce the following notion of tame parcels and tame factors to discuss asymptotic critical points by polynomials.

Definition 13.10. *Assume $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ and an infinite-length fitting path $P = ((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$.*

1. *We call $\mathscr{F}$ tame along P by $t \in \mathbb{Q}(\mathfrak{X})$ if $t \geq_{\mathrm{O}_{\mathfrak{X}}} 0$ and*

$$t(r) = \lim_{i\to\infty} \frac{f_{s,n_i\boxplus k_i}}{f_{s,n_i}}(r)$$

for each $r \in \mathrm{O}_{\mathfrak{X}}$. We call this t the tame factor of $\mathscr{F}$ along P.

2. *Let $\mathscr{F}$ be tame along P by $t \in \mathbb{Q}(\mathfrak{X})$. Then, let* $\mathrm{AIr}(\mathfrak{X})(\zeta), \mathrm{AIr}(\mathfrak{X})(t,\zeta) \in \mathbb{Q}[\mathfrak{X}]^2$ *such that*

$$\mathrm{AIr}(\mathfrak{X})(\zeta) = \mathrm{Ir}(\mathfrak{X})(\prod \phi(x)^{\sigma(\kappa)^\vee \circ w}),$$
$$\mathrm{AIr}(\mathfrak{X})(t,\zeta) = \mathrm{Ir}(\mathfrak{X})(t \cdot \prod \psi(x)^{\sigma(\kappa)^\vee \circ w}).$$

Let us state the following polynomials of rational functions by squaring orders.

Lemma 13.11. *For $f \in \mathbb{Q}(\mathfrak{X})$, consider* $\mathrm{Ir}(\mathfrak{X})(f) \in \mathbb{Q}[\mathfrak{X}]^2$.

1. *$f \geq_{\mathrm{O}_\mathfrak{X}} 0$ if and only if there exists $f(r) = \mathrm{Frac}(\mathrm{Ir}(\mathfrak{X})(f)(r)) \in \mathbb{R}_{\geq 0}$ for each $r \in \mathrm{O}_\mathfrak{X}$.*
2. *$f >_{\mathrm{O}_\mathfrak{X}} 0$ if and only if there exists $f(r) = \mathrm{Frac}(\mathrm{Ir}(\mathfrak{X})(f)(r)) \in \mathbb{R}_{> 0}$ for each $r \in \mathrm{O}_\mathfrak{X}$.*
3. *Suppose a squaring order $\succ$ on $\mathfrak{X}$. If $f^2 \succ 0$, then $(\prod \mathrm{Ir}(\mathfrak{X})(f))(r) \neq 0$ for each $r \in \mathrm{O}_\mathfrak{X}$.*

Proof. Proof of Claim 1. Assume $f \geq_{\mathrm{O}_\mathfrak{X}} 0$. Then, $\mathrm{Ir}(\mathfrak{X})(f)_1(r) \neq 0$ for each $r \in \mathrm{O}_\mathfrak{X}$, since $f(r) \in \mathbb{R}$ must exist. Thus, the only if part holds. The if part is clear.

Proof of Claim 2. We obtain Claim 2 similarly, replacing $\geq$ with $>$.

Proof of Claim 3. Claim 3 follows from Claim 2, since $f^2 = \frac{\mathrm{Ir}(\mathfrak{X})(f)_2^2}{\mathrm{Ir}(\mathfrak{X})(f)_1^2} >_{\mathrm{O}_\mathfrak{X}} 0$ by $>_{\mathrm{O}_\mathfrak{X}} \Supset \succ$. ☐

On merged pairs and almost strictly unimodal sequences, we then prove the following limit properties, which are independent to mediators.

Proposition 13.12. *Consider a merged pair $\zeta = (P, \mathscr{F})$ with $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ and an infinite-length $P = ((s,l,m_i,n_i,k_i))_{i \in [\![\theta]\!]}$. Let ψ be the l-canonical mediator and $\kappa = k_{\theta_1}$.*

1. *Suppose $\lim_{j\to\infty} \frac{f_{s,n_j \boxplus k_j}}{f_{s,n_j}}(r) = 0$ for some $r \in \mathrm{O}_\mathfrak{X}$. Then, we have*

$$\lim_{j\to\infty} \frac{\mathscr{F}_{n_j \boxplus k_j}}{\mathscr{F}_{n_j}}(r) = 0. \tag{13.3.1}$$

   In particular, $u(\zeta, r)$ is a hill or decreasing sequence such that

$$\lim_{i\to\infty} u(\zeta, r)_i = 0. \tag{13.3.2}$$

2. *For each $r \in \mathrm{O}_\mathfrak{X}$, $\lim_{j\to\infty} \frac{f_{s,n_j \boxplus k_j}}{f_{s,n_j}}(r) = \infty$ does not hold.*
3. *Let $\mathscr{F}$ be tame along P by $t \in \mathbb{Q}(\mathfrak{X})$ and $r \in \mathrm{O}_\mathfrak{X}$. Then, we have*
   (a) $\lim_{j\to\infty} \frac{\mathscr{F}_{n_j \boxplus k_j}}{\mathscr{F}_{n_j}}(r) = t(r) \cdot \frac{\prod \psi(x(r))^{\sigma(\kappa)^\vee \circ w}}{\prod \phi(x(r))^{\sigma(\kappa)^\vee \circ w}} \in \mathbb{R}_{\geq 0}$;
   (b) $\det(\mathrm{AIr}(\mathfrak{X})(\zeta), \mathrm{AIr}(\mathfrak{X})(t,\zeta))(r) = 0$ *if and only if* $\lim_{j\to\infty} \frac{\mathscr{F}_{n_j \boxplus k_j}}{\mathscr{F}_{n_j}}(r) = 1$.

Proof. Proof of Claim 1. First, we obtain equation (13.3.1) from Claim 1 of Proposition 12.12 and Lemma 13.9. Second, we obtain equation (13.3.2) as follows. The path-parcel sequence $u(\zeta, r)$ is almost strictly unimodal by Theorem 12.18, and of positive real numbers by Definition 13.4. Therefore, equation (13.3.1) gives $\lambda \in [\![\theta]\!]$ such that $\left(u(\zeta,r)_j = \mathscr{F}_{n_j}(r)\right)_{j\geq\lambda}$ is strictly decreasing. This gives $v \in \mathbb{R}_{\geq 0}$ such that $\lim_{j\to\infty} u(\zeta,r)_j = v$. If $v > 0$, then $\lim_{j\to\infty} \frac{u(\zeta,r)_{j+1}}{u(\zeta,r)_j} = 1$ against equation (13.3.1).

Proof of Claim 2. Let us assume otherwise. Then, Lemma 13.9 and Claim 1 of Proposition 12.12 imply $\lim_{j\to\infty} \frac{u(\zeta,r)_{j+1}}{u(\zeta,r)_j} = \infty$. However, because $u(\zeta,r)$ is strictly log-concave by Theorem 12.18, each $j \in [\![\theta]\!]$ satisfies $\infty > \frac{u(\zeta,r)_{j+1}}{u(\zeta,r)_j} > \frac{u(\zeta,r)_{j+2}}{u(\zeta,r)_{j+1}} > \dots$.

Proof of Claim 3a. Since $t \geq_{\mathrm{O}_{\mathfrak{X}}} 0$, Claim 3a holds by Lemma 13.9 and Claim 1 of Proposition 12.12.

Proof of Claim 3b. Claim 1 of Proposition 12.12 gives $\prod \phi(x)^{\sigma(\kappa)^{\vee}\circ w} >_{\mathrm{O}_{\mathfrak{X}}} 0$. Thus, Claim 3a implies

$$\lim_{j\to\infty} \frac{\mathscr{F}_{n_j \boxplus k_j}}{\mathscr{F}_{n_j}}(r) = 1$$

if and only if

$$t(r) \cdot \prod \psi(x(r))^{\sigma(\kappa)^{\vee}\circ w} = \prod \phi(x(r))^{\sigma(\kappa)^{\vee}\circ w}.$$

Since $t \cdot \prod \psi(x)^{\sigma(\kappa)^{\vee}\circ w} \geq_{\mathrm{O}_{\mathfrak{X}}} 0$ by Claim 1 of Proposition 12.12, we obtain Claim 3b by Claims 1 and 2 of Lemma 13.11. □

We introduce the following notation.

Definition 13.13. *Assume $\lambda \in \mathbb{Z}_{\geq 1}$, $h \in \mathbb{Z}$, $l = 1$, and $s = (0,\infty)$. Then, let*

$$\pi(\lambda,h) = \mathbb{P}_{s,l,\iota^{\lambda}(1),h}.$$

Let $\pi(\lambda) = \pi(\lambda,0)$ for simplicity.

We then have the following explicit description of $\pi(\lambda,h)$.

Lemma 13.14. *Let $\lambda \in \mathbb{Z}_{\geq 1}$, $h \in \mathbb{Z}$, $l = 1$, $s = (0,\infty)$, and $\theta = (1,\infty)$.*

1. *We have $\pi(\lambda,h) = ((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$ with $a_i = \nu(k_i)$ and $b_i = \nu(m_i,n_i,k_i)$ such that each $i \in [\![\theta]\!]$ satisfies*

$$\begin{aligned} a_i = k_i &= (0,\lambda) \in \mathbb{Z}^{2l}, \\ b_i &= (\lambda i + h, \lambda(i+1) + h) \in \mathbb{Z}^{2l}, \\ m_i = n_i &= (\lambda i + h) \in \mathbb{Z}^{l}. \end{aligned}$$

2. *We have $m_1 \boxminus k_1 = (h) \in \mathbb{Z}^l$.*

3. *$\pi(\lambda,h)$ is wrapped if and only if $h \in \mathbb{Z}_{\geq 0}$. In particular, $\pi(\lambda)$ is wrapped.*

Proof. Proof of Claim 1. We have $\iota^{\lambda}(1) \in Y(\lambda,l)$ and $a_i = \nu(k_i) = k_i$ by $a_{i,1} = 0$. Thus, Claim 1 holds by Theorem 12.32, since

$$\begin{aligned} z(l,0,\iota^{\lambda}(1)) &= \left(\mathrm{L}_0(\iota^{\lambda}(1))\right) = (0), \\ \mathrm{L}(l,\iota^{\lambda}(1)) &= (\lambda). \end{aligned}$$

Proof of Claim 2. Claim 2 follows from $\sigma(k_i) = (\lambda)$.

Proof of Claim 3. We obtain Claim 3, because $\pi(\lambda,h)$ is wrapped if and only if $(h) \in [\![0,\infty]\!]^l$ by Claim 2, and $\pi(\lambda) = \pi(\lambda,0)$. □

We introduce the notation below for $\pi(\lambda)$ and parcels.

Definition 13.15. *Suppose a parcel $\mathscr{F}=\Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ and $\lambda\in\mathbb{Z}_{\geq 1}$. Then, we define the tuple*

$$\Omega_\lambda(\mathscr{F})=(\pi(\lambda),\mathscr{F}).$$

In particular, let $\Omega(\mathscr{F})=\Omega_1(\mathscr{F})$.

Then, we obtain the following vanishing constraint on parcel numerators.

Proposition 13.16. *Let $p\in\mathbb{Z}$. For each $h\in\mathbb{Z}_{\geq 0}$, suppose $u_h\in\mathbb{Z}_{\geq 2}$, $v_h\in\mathbb{Z}_{\leq -1}$, $\lambda_{h,0,0}\in\mathbb{Z}$, and finitely many non-zero $\lambda_{h,i,j}\in\mathbb{Z}$ of $i\in\mathbb{Z}_{\geq 1}$ and $j\in\mathbb{Z}_{\neq 0}$ with the following conditions:*

1. *$i\leq u_h$ and $j\geq v_h$ if $i\in\mathbb{Z}_{\geq 1}$ and $j\in\mathbb{Z}_{\leq -1}$ satisfy $\lambda_{h,i,j}\neq 0$;*
2. *$u_{h+1}\geq u_h$ and $v_{h+1}\leq v_h$;*
3. *$\lim_{h\to\infty}\lambda_{h,u_h,v_h}=p$;*
4. *$\lim_{h\to\infty}\sum_{i\in\mathbb{Z}_{\geq 1},j\in\mathbb{Z}_{\neq 0},(i,j)\neq(u_h,v_h)}\left|\lambda_{h,i,j}\right|y^h=\lim_{h\to\infty}\lambda_{h,0,0}y^h=0$ if $0<y<1$.*

Let $s=(0,\infty)$, $l=1$, and q be $\succ$-admissible. Consider a $\succ'$-merged-log-concave $\mathscr{F}=\Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ such that each $(h)\in[\![s]\!]^l$ satisfies

$$f_{s,(h)}=\lambda_{h,0,0}+\sum_{i\in\mathbb{Z}_{\geq 1},j\in\mathbb{Z}_{\neq 0}}\lambda_{h,i,j}q^{jh^i}\in\mathbb{Q}(\mathfrak{X}).$$

Then, we obtain

$$\lambda_{h,u_h,v_h}=0$$

for infinitely many $h\in\mathbb{Z}_{\geq 0}$.

Proof. Suppose $r\in\mathrm{O}_{\mathfrak{X}}$ and $h\in\mathbb{Z}_{\geq 2}$. First, we prove

$$\lim_{h\to\infty}\lambda_{h,0,0}q(r)^{-v_hh^{u_h}}=0. \tag{13.3.3}$$

Claim 1 of Lemma 5.22 gives

$$0<q(r)<1. \tag{13.3.4}$$

Also, since $-v_h\in\mathbb{Z}_{\geq 1}$, each $i\in\mathbb{Z}_{\geq 1}$ satisfies

$$-v_hh^i\geq h. \tag{13.3.5}$$

Thus, inequalities (13.3.4) and (13.3.5) and Assumption 4 give equation (13.3.3).

Second, let $p(i,j,h)=jh^i-v_hh^{u_h}$ for $i\in\mathbb{Z}_{\geq 1}$ and $j\in\mathbb{Z}_{\neq 0}$. When $\lambda_{h,i,j}\neq 0$ and $(i,j)\neq(u_h,v_h)$, we prove

$$p(i,j,h)\geq h. \tag{13.3.6}$$

Assumption 1 gives the following three cases:

- if $j \in \mathbb{Z}_{\geq 1}$, then inequality (13.3.5) implies $p(i,j,h) \geq -v_h h^{u_h} \geq h$;
- if $j \in [\![v_h, -1]\!]$ and $i \in [\![u_h - 1]\!]$, then $h^{u_h - i} - 1 \geq 1$ and inequality (13.3.5) imply $p(i,j,h) \geq v_h h^i - v_h h^{u_h} = -v_h h^i (h^{u_h - i} - 1) \geq -v_h h^i \geq h$;
- if $j \in [\![v_h + 1, -1]\!]$ and $i = u_h$, then $j - v_h \geq 1$ yields $p(i,j,h) = h^{u_h}(j - v_h) \geq h$.

Third, let $\pi(1) = ((s,l,m_i,n_i,k_i))_{i \in [\![\theta]\!]}$ with $U(h,q) = q^{-v_h h^{u_h}} f_{s,n_h}$ and $V(h,q) = U(h,q) - \lambda_{h,0,0} q^{-v_h h^{u_h}} - \lambda_{h,u_h,v_h}$. Then, Assumption 4 yields

$$\lim_{h \to \infty} V(h, q(r)) = 0, \tag{13.3.7}$$

because Claim 1 of Lemma 13.14 and inequalities (13.3.4) and (13.3.6) give

$$|V(h,q(r))| \leq \sum_{i \in \mathbb{Z}_{\geq 1}, j \in \mathbb{Z}_{\neq 0}, (i,j) \neq (u_h, v_h)} \left| \lambda_{h,i,j} q(r)^{p(i,j,h)} \right| \leq \sum_{i \in \mathbb{Z}_{\geq 1}, j \in \mathbb{Z}_{\neq 0}, (i,j) \neq (u_h, v_h)} \left| \lambda_{h,i,j} \right| q(r)^h.$$

Finally, we prove the assertion by contradiction. Assumption 3 implies $\lim_{h \to \infty} \lambda_{h,u_h,v_h} = p \neq 0$. We deduce $\lim_{h \to \infty} U(h,q(r)) = p \neq 0$ by equations (13.3.3) and (13.3.7). In particular,

$$\lim_{h \to \infty} \frac{U(h+1, q(r))}{U(h, q(r))} = 1. \tag{13.3.8}$$

Also, inequality (13.3.5) and Assumption 2 imply

$$v_h h^{u_h} - v_{h+1}(h+1)^{u_{h+1}} \geq v_h h^{u_h} - v_h (h+1)^{u_h} \geq -v_h u_h h^{u_h - 1} \geq h,$$

since $-v_h(h+1)^{u_h} = -v_h h^{u_h} - v_h u_h h^{u_h - 1} - \dots$. Then, inequality (13.3.4) yields

$$\lim_{h \to \infty} \frac{q(r)^{v_{h+1}(h+1)^{u_{h+1}}}}{q(r)^{v_h h^{u_h}}} = \lim_{h \to \infty} \frac{1}{q(r)^{v_h h^{u_h} - v_{h+1}(h+1)^{u_{h+1}}}} = \infty.$$

Therefore, because $\frac{f_{s,n_h \boxplus k_h}}{f_{s,n_h}} = \frac{f_{s,n_{h+1}}}{f_{s,n_h}}$, equation (13.3.8) gives

$$\lim_{h \to \infty} \frac{f_{s,n_h \boxplus k_h}}{f_{s,n_h}}(r) = \lim_{h \to \infty} \frac{q(r)^{v_{h+1}(h+1)^{u_{h+1}}}}{q(r)^{v_h h^{u_h}}} \cdot \lim_{h \to \infty} \frac{U(h+1, q(r))}{U(h, q(r))} = \infty.$$

This contradicts Claim 2 in Proposition 13.12, since $\Omega(\mathscr{F})$ is a merged pair by Claim 3 of Lemma 13.14. □

Also, we obtain the following vanishing constraint on parcel numerators without assuming t-monomials (see Remark 8.38). Compared to Proposition 13.16, the following allows $j \in \mathbb{Q}_{\neq 0}$ and takes $\lambda_{i,j} \in \mathbb{Q}$, which is independent to $m \in \mathbb{Z}^l$.

Proposition 13.17. *Assume finitely many non-zero $\lambda_{0,0}, \lambda_{i,j} \in \mathbb{Q}$ for $i \in \mathbb{Z}_{\geq 1}$ and $j \in \mathbb{Q}_{\neq 0}$. Let $s = (0, \infty)$, $l = 1$, and q be $\succ$-admissible. Consider a $\succ'$-merged-log-concave $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ such that each $(h) \in [\![s]\!]^l$ satisfies*

$$f_{s,(h)} = \lambda_{0,0} + \sum_{i \in \mathbb{Z}_{\geq 1}, j \in \mathbb{Q}_{\neq 0}} \lambda_{i,j} q^{j h^i} \in \mathbb{Q}(\mathfrak{X}).$$

Then, we have

$$\lambda_{i,j} = 0$$

for each $i \in \mathbb{Z}_{\geq 2}$ and $j \in \mathbb{Q}_{<0}$.

Proof. We prove the assertion by contradiction. Hence, suppose the largest $u \in \mathbb{Z}_{\geq 2}$ such that there exists $j \in \mathbb{Q}_{<0}$ with $\lambda_{u,j} \neq 0$, and the smallest $v \in \mathbb{Q}_{<0}$ such that $\lambda_{u,v} \neq 0$. Let $\pi(1) = ((s,l,m_i,n_i,k_i))_{i \in [\![\theta]\!]}$, $h \in \mathbb{Z}_{\geq 2}$, and $r \in \mathrm{O}_{\mathfrak{X}}$.

First, suppose $p(i,j,h) = jh^i - vh^u$ for $i \in \mathbb{Z}_{\geq 1}$ and $j \in \mathbb{Q}_{\neq 0}$. Then, by Claim 1 of Lemma 13.14, we consider

$$U(h,q) = q^{-vh^u} f_{s,n_h} = \lambda_{0,0} q^{-vh^u} + \lambda_{u,v} + \sum_{i \in \mathbb{Z}_{\geq 1}, j \in \mathbb{Q}_{\neq 0}, (i,j) \neq (u,v)} \lambda_{i,j} q^{p(i,j,h)}.$$

Since $u \in \mathbb{Z}_{\geq 2}$ and $v \in \mathbb{Q}_{<0}$, $\lim_{h\to\infty} -vh^u = \infty$. Hence, if $\lambda_{i,j} \neq 0$ and $(i,j) \neq (u,v)$, then $\lim_{h\to\infty} p(i,j,h) = \infty$, as we have the only three cases: $j > 0$; $0 > j > v$ and $i = u$; and, $0 > j$ and $1 \leq i < u$. In particular, since $0 < q(r) < 1$ by Claim 1 of Lemma 5.22, we have

$$\lim_{h\to\infty} U(h,q(r)) = \lambda_{u,v}. \tag{13.3.9}$$

Second, $u \in \mathbb{Z}_{\geq 2}$ and $v \in \mathbb{Q}_{<0}$ imply $vh^u - v(h+1)^u \geq -vuh^{u-1} \geq -vh > 0$, since $-v(h+1)^u = -vh^u - vuh^{u-1} - \dots$. Hence, we have

$$\lim_{h\to\infty} \frac{q(r)^{v(h+1)^u}}{q(r)^{vh^u}} = \lim_{h\to\infty} \frac{1}{q(r)^{vh^u - v(h+1)^u}} = \infty.$$

Then, since $\lambda_{u,v} \neq 0$, equation (13.3.9) yields

$$\lim_{h\to\infty} \frac{f_{s,n_h \boxplus k_h}}{f_{s,n_h}}(r) = \lim_{h\to\infty} \frac{q(r)^{v(h+1)^u}}{q(r)^{vh^u}} \cdot \lim_{h\to\infty} \frac{U(h+1,q(r))}{U(h,q(r))} = \infty.$$

This contradicts Claim 2 in Proposition 13.12, since $\Omega(\mathscr{F})$ is a merged pair by Claim 3 of Lemma 13.14. □

13.4 Merged pairs and critical points

By front, rear, and asymptotic critical points of merged pairs, we discuss the variation of almost strictly unimodal sequences. Also, we obtain real algebraic sets not only of front and rear critical points, but also of asymptotic critical points by tame factors. Moreover, we introduce the notion of semi-phase transitions and phase transitions by these critical points. In particular, we obtain front phase transitions by monomial parcels of general lengths.

First, we introduce the notion of path-parcel differences.

Definition 13.18. *Suppose a $(\theta,\mathfrak{X})$-merged pair $\zeta = (P,\mathscr{F})$ such that $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ and $P = ((s,l,m_i,n_i,k_i))_{i \in [\![\theta]\!]}$. Assume the canonical l-mediator ψ. Then, we define the following rational functions:*

1. $\mathrm{F_D}(\zeta) = \mathscr{F}_{m_{\theta_1} \boxminus k_{\theta_1}} - \mathscr{F}_{n_{\theta_1}}$*;*
2. $\mathrm{R_D}(\zeta) = \mathscr{F}_{n_{\theta_2}} - \mathscr{F}_{n_{\theta_2} \boxplus k_{\theta_2}}$*, if P is finite-length;*
3. $\mathrm{A_D}(\zeta) = \prod \phi(x)^{\sigma(\kappa)^\vee \circ w} - t \cdot \prod \psi(x)^{\sigma(\kappa)^\vee \circ w}$*, if $\mathscr{F}$ is tame along P by $t \in \mathbb{Q}(\mathfrak{X})$.*

We call $\mathrm{F_D}(\zeta), \mathrm{R_D}(\zeta)$*, and* $\mathrm{A_D}(\zeta)$ *front, rear, and asymptotic path-parcel differences.*

Then, we obtain the following variation of almost strictly unimodal sequences.

Theorem 13.19. *Suppose a $(\theta, \mathfrak{X})$-merged pair $\zeta = (P, \mathscr{F})$ with $r \in \mathrm{O}_{\mathfrak{X}}$.*

1. *Then, we have the following.*
   (a) *$u(\zeta, r)$ is a strictly decreasing sequence if $\mathrm{F_D}(\zeta)(r) > 0$.*
   (b) *$u(\zeta, r)$ is a hill and decreasing sequence if $\mathrm{F_D}(\zeta)(r) = 0$.*
   (c) *$u(\zeta, r)$ is a two-sided hill or strictly increasing sequence if $\mathrm{F_D}(\zeta)(r) < 0$.*
2. *Suppose that P is finite-length. Then, we have the following.*
   (a) *$u(\zeta, r)$ is a two-sided hill or strictly decreasing sequence if $\mathrm{R_D}(\zeta)(r) > 0$.*
   (b) *$u(\zeta, r)$ is a hill and increasing sequence if $\mathrm{R_D}(\zeta)(r) = 0$.*
   (c) *$u(\zeta, r)$ is a strictly increasing sequence if $\mathrm{R_D}(\zeta)(r) < 0$.*
3. *Suppose that $\mathscr{F}$ is tame along P. Then, we have the following.*
   (a) *$u(\zeta, r)$ is a two-sided hill or strictly decreasing sequence if $\mathrm{A_D}(\zeta)(r) > 0$.*
   (b) *$u(\zeta, r)$ is an asymptotic hill if $\mathrm{A_D}(\zeta)(r) = 0$.*
   (c) *$u(\zeta, r)$ is a strictly increasing sequence if $\mathrm{A_D}(\zeta)(r) < 0$.*

Proof. Let $P = ((s, l, m_i, n_i, k_i))_{i \in [\![\theta]\!]}$.

Proof of Claim 1. The first and second terms of $u(\zeta, r)$ are $\mathscr{F}_{m_{\theta_1} \boxminus k_{\theta_1}}(r)$ and $\mathscr{F}_{n_{\theta_1}}(r)$. Thus, Claim 1a holds by Claim 1 of Proposition 13.3, because $u(\zeta, r)$ is almost strictly unimodal by Theorem 12.18. Claims 1b and 1c hold similarly.

Proof of Claim 2. The penultimate and last terms of $u(\zeta, r)$ are $\mathscr{F}_{n_{\theta_2}}(r)$ and $\mathscr{F}_{n_{\theta_2} \boxplus k_{\theta_2}}(r)$. Thus, Claim 2a holds by Claim 2 of Proposition 13.3 and Theorem 12.18. We obtain Claims 2b and 2c similarly.

Proof of Claim 3. Claims 3a and 3c follow from Claim 1 of Proposition 12.12, Claim 3a of Proposition 13.12, and Theorem 12.18. Claim 3b holds by Proposition 13.7. □

Second, we state the following lemma for the real algebraic sets of critical points.

Lemma 13.20. *Consider $\mathscr{F} = \Lambda(s, l, w, \succ, f_s, \phi, \rho, x, \mathfrak{X})$. Let $r \in \mathrm{O}_{\mathfrak{X}}$ and $m, n \in [\![s]\!]^l$. Then, we have the following.*

1. $\mathrm{Ir}(\mathfrak{X})(\mathscr{F}_m)_1(r) \neq 0$.
2. $(\mathscr{F}_m - \mathscr{F}_n)(r) = 0$ *if and only if* $\det(\mathrm{Ir}(\mathfrak{X})(\mathscr{F}_m), \mathrm{Ir}(\mathfrak{X})(\mathscr{F}_n))(r) = 0$.

Proof. Proof of Claim 1. Since f_s is $\succ$-positive, we have $f_{s,m}^2 \succ 0$. This implies $f_{s,m}^2 >_{\mathrm{O}_{\mathfrak{X}}} 0$. Also, $\prod \phi(x)^{m \circ w} \cdot [m]!_x^w >_{\mathrm{O}_{\mathfrak{X}}} 0$ by Proposition 12.12. Then, $\mathscr{F}_m^2 >_{\mathrm{O}_{\mathfrak{X}}} 0$ by $\mathscr{F}_m = \frac{f_{s,m}}{\prod \phi(x)^{m \circ w} \cdot [m]!_x^w}$ and Lemma 12.11. Hence, we obtain Claim 1 by Claim 3 of Lemma 13.11.

Proof of Claim 2. Claim 2 follows from Claim 1. □

We introduce the following real algebraic sets.

Definition 13.21. *Suppose a $(\theta, \mathfrak{X})$-merged pair $\zeta = (P, \mathscr{F})$ such that $P = ((s, l, m_i, n_i, k_i))_{i \in [\![\theta]\!]}$. Then, we define the following:*

1. $\mathrm{F_Z}(\zeta) = \left\{ r \in \mathrm{O}_{\mathfrak{X}} \mid \det(\mathrm{Ir}(\mathfrak{X})(\mathscr{F}_{m_{\theta_1} \boxminus k_{\theta_1}}), \mathrm{Ir}(\mathfrak{X})(\mathscr{F}_{n_{\theta_1}}))(r) = 0 \right\}$;

2. $\mathrm{R_Z}(\zeta) = \left\{ r \in \mathrm{O}_{\mathfrak{X}} \mid \det(\mathrm{Ir}(\mathfrak{X})(\mathscr{F}_{n_{\theta_2}}), \mathrm{Ir}(\mathfrak{X})(\mathscr{F}_{n_{\theta_2} \boxplus k_{\theta_2}}))(r) = 0 \right\}$, *if P is finite-length;*

3. $\mathrm{A_Z}(\zeta) = \{ r \in \mathrm{O}_{\mathfrak{X}} \mid \det(\mathrm{AIr}(\mathfrak{X})(\zeta), \mathrm{AIr}(\mathfrak{X})(t, \zeta))(r) = 0 \}$ *if $\mathscr{F}$ is tame along P by $t \in \mathbb{Q}(\mathfrak{X})$.*

By Claim 1 of Lemma 13.11, $\mathrm{F_Z}(\zeta)$ above ignores the choices of $\mathrm{Ir}(\mathfrak{X})(\mathscr{F}_{m_{\theta_1} \boxminus k_{\theta_1}}), \mathrm{Ir}(\mathfrak{X})(\mathscr{F}_{n_{\theta_1}}) \in \mathbb{Q}[\mathfrak{X}]$. The same holds for $\mathrm{R_Z}(\zeta)$ and $\mathrm{A_Z}(\zeta)$.

We thus have the real algebraic sets of critical points.

Theorem 13.22. *Suppose a $(\theta, \mathfrak{X})$-merged pair $\zeta = (P, \mathscr{F})$. Then, $\mathrm{F_Z}(\zeta)$, $\mathrm{R_Z}(\zeta)$, and $\mathrm{A_Z}(\zeta)$ are the real algebraic sets of front, rear, and asymptotic critical points of ζ, respectively. Namely, $r \in \mathrm{O}_{\mathfrak{X}}$ is in $\mathrm{F_Z}(\zeta)$, $\mathrm{R_Z}(\zeta)$, or $\mathrm{A_Z}(\zeta)$ if and only if $\mathrm{F_D}(\zeta)(r) = 0$, $\mathrm{R_D}(\zeta)(r) = 0$, or $\mathrm{A_D}(\zeta)(r) = 0$, respectively.*

Proof. Statements hold by Claim 2 of Lemma 13.20, Claim 3b of Proposition 13.12, and Theorem 13.19. □

Third, we introduce the notion of front, rear, and asymptotic semi-phase and phase transitions of merged pairs. In Section 14.3, there is a merged pair with a semi-phase transition but not with a phase transition. Also, in Section 14.4, there is a merged pair with asymptotic critical points but not with a semi-phase transition.

Definition 13.23. *Suppose a $(\theta, \mathfrak{X})$-merged pair $\zeta = (P, \mathscr{F})$.*

1. *We say that ζ has a front, rear, or asymptotic semi-phase transition if*
   - $\emptyset \neq \mathrm{F_Z}(\zeta) \neq \mathrm{O}_{\mathfrak{X}}$,
   - $\emptyset \neq \mathrm{R_Z}(\zeta) \neq \mathrm{O}_{\mathfrak{X}}$, *or*
   - $\emptyset \neq \mathrm{A_Z}(\zeta) \neq \mathrm{O}_{\mathfrak{X}}$, *respectively.*

2. *We say that ζ has a front, rear, or asymptotic phase transition between r_1 and r_2 if*
   - $\mathrm{F_D}(\zeta)(r_1) \cdot \mathrm{F_D}(\zeta)(r_2) < 0$,
   - $\mathrm{R_D}(\zeta)(r_1) \cdot \mathrm{R_D}(\zeta)(r_2) < 0$, *or*
   - $\mathrm{A_D}(\zeta)(r_1) \cdot \mathrm{A_D}(\zeta)(r_2) < 0$, *respectively.*

Remark 13.24. Strictly decreasing sequences turn into two-sided hill or strictly increasing sequences by front phase transitions, which give statistical-mechanical phase transitions in Sections 1.11 and 19.

We obtain front phase transitions on monomial parcels by the lemma below.

Lemma 13.25. *Let $T \in \mathfrak{X}$. Assume $\mathrm{Bot}(T), \mathrm{Top}(T) \in \mathbb{Q}(\mathfrak{X})^2$ with the following conditions:*

1. $\mathrm{Bot}_1(T), \mathrm{Bot}_2(T), \mathrm{Top}_1(T), \mathrm{Top}_2(T) >_{\mathrm{O}_{\mathfrak{X}}} 0$;
2. $\mathrm{Frac}(\mathrm{Bot})(0) > \mathrm{Frac}(\mathrm{Top})(0)$ *in* $\mathbb{R}$;
3. $\mathrm{Frac}(\mathrm{Bot})(1) < \mathrm{Frac}(\mathrm{Top})(1)$ *in* $\mathbb{R}$.

Also, let $F_i(T) = \frac{\mathrm{Top}_i(T)}{\mathrm{Bot}_i(T)}$ for $i \in [\![2]\!]$. Then, there are real numbers $0 < r_0 < r_1 < r_2 < 1$ such that $F_1(r_0) < F_2(r_0)$, $F_1(r_1) = F_2(r_1)$, and $F_1(r_2) > F_2(r_2)$.

Proof. Conditions 2 and 3 give $0 < r_0 < r_1 < r_2 < 1$ such that $\mathrm{Frac}(\mathrm{Bot})(r_0) > \mathrm{Frac}(\mathrm{Top})(r_0)$, $\mathrm{Frac}(\mathrm{Bot})(r_1) = \mathrm{Frac}(\mathrm{Top})(r_1)$, and $\mathrm{Frac}(\mathrm{Bot})(r_2) < \mathrm{Frac}(\mathrm{Top})(r_2)$. Now, Condition 1 implies the assertion. □

Proposition 13.26. *Let (l,w,γ) be a monomial index with palindromic w, $(\gamma_{i,1})_{i\in[\![l]\!]}$, and $(\gamma_{i,2})_{i\in[\![l]\!]}$. For $s=(0,\infty)$, assume a wrapped fitting path $P=((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$ and monomial parcel $\mathscr{F}=\Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\rho,x,\mathfrak{X})$. Let $\zeta=(P,\mathscr{F})$. Then, we have the following.*

1. *ζ is a merged pair.*
2. *If q is fully admissible by $\mathfrak{X}$, then the following statements are equivalent.*
   (a) *ζ has a front semi-phase transition;*
   (b) *$t_\gamma(m_{\theta_1})-t_\gamma(m_{\theta_1}\boxminus k_{\theta_1})\in\mathbb{Q}_{>0}$;*
   (c) *ζ has a front phase transition.*

   Moreover, if $\mathfrak{X}=\{X_1\}$ and $t_\gamma(m_{\theta_1})-t_\gamma(m_{\theta_1}\boxminus k_{\theta_1})\in\mathbb{Q}_{>0}$, then ζ has a front phase transition at the unique front critical point.

Proof. Let $\theta_1=1$ and each $\gamma_{i,3}=0$ for simplicity. Let $r\in\mathrm{O}_{\mathfrak{X}}$.

Proof of Claim 1. First, $\mathscr{F}$ is flip-invariant by Proposition 12.8. Second, $\mathscr{F}$ is $>_{\mathrm{O}_{\mathfrak{X}}}$-merged-log-concave by Theorem 8.40.

Proof of Claim 2. Let $T\in\mathfrak{X}$ and $h\in\mathbb{Z}_{\geq 1}$ such that $T=q^{\frac{1}{h}}$ for the fully admissible q. First, we prove Statement 2b from Statement 2a. Since P is wrapped, $m_1\boxminus k_1\in[\![s]\!]^l$. Also, since $\mathscr{F}$ is flip-invariant, $\mathscr{F}_{n_1}(r)=\mathscr{F}_{m_1}(r)$. Thus, there exists $c\in\mathrm{O}_{\mathfrak{X}}$ such that

$$\mathscr{F}_{m_1\boxminus k_1}(c)=\frac{q(c)^{t_\gamma(m_1\boxminus k_1)}}{(m_1\boxminus k_1)_q^w|_{q=q(c)}}=\frac{q(c)^{t_\gamma(m_1)}}{(m_1)_q^w|_{q=q(c)}}=\mathscr{F}_{n_1}(c). \tag{13.4.1}$$

Moreover, we obtain $0<\frac{(m_1)_q^w|_{q=q(c)}}{(m_1\boxminus k_1)_q^w|_{q=q(c)}}<1$ by $\sigma(k_1)>0$ and $w>0$. Hence, equation (13.4.1) implies Statement 2b, since otherwise $q(c)^{t_\gamma(m_{\theta_1})-t_\gamma(m_{\theta_1}\boxminus k_{\theta_1})}\geq 1$.

Second, we prove Statement 2c from Statement 2b. Let $\mathrm{Bot}_1(T)=(m_1\boxminus k_1)_q^w$, $\mathrm{Bot}_2(T)=(m_1)_q^w$, $\mathrm{Top}_1=q^{t_\gamma(m_1\boxminus k_1)}$, and $\mathrm{Top}_2(T)=q^{t_\gamma(m_1)}\in\mathbb{Q}(\mathfrak{X})$. Then, Lemma 13.25 gives a front phase transition of ζ.

Third, Statement 2c implies Statement 2a by Definition 13.23. The latter statement holds by $\mathfrak{X}=\{T\}$. □

In Sections 13.7 and 17.2, we further discuss phase transitions by width-one monomial parcels.

13.5 Ideal merged pairs

There is a merged pair $(P,\mathscr{F})$ such that $\mathscr{F}$ is not $>_q$-merged-log-concave, but its merged determinants along P give q-polynomials (see Section 14.6.2). We introduce the following notion of ideal merged pairs to obtain not only almost strictly unimodal sequences, but also polynomials with positive integer coefficients. This extends Definition 1.18.

Definition 13.27. *Suppose a $(\theta,\mathfrak{X})$-merged pair $\zeta=(P,\mathscr{F})$ such that $P=((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$ and $\mathscr{F}=\Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$. We call ζ ideal if each $i\in[\![\theta]\!]$ satisfies*

$$\Delta(\mathscr{F})(s,l,w,m_i,n_i,k_i,\phi,\rho,x,\mathfrak{X})>_x 0.$$

For example, suppose an ideal $(\theta,\mathfrak{X})$-merged pair $\zeta=(P,\mathscr{F})$ such that $\mathscr{F}=\Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ and $x=(q)$. Then, each $i\in[\![\theta]\!]$ satisfies $\Delta(\mathscr{F})(s,l,w,m_i,n_i,k_i,\phi,\rho,x,\mathfrak{X})>_q 0$. Also, these q-polynomials give almost strictly unimodal sequences $u(\zeta,r)$ in Theorems 12.18, 13.19, and 13.22. We compute several explicit ideal merged pairs with critical points in Section 14.

13.6 Comparison of fitting paths

We introduce the following notions of finer and equivalent fitting paths. By the notions, we compare wrapped fitting paths, which give almost strictly unimodal sequences by merged pairs. In particular, we discuss the existence and the non-existence of finest wrapped fitting paths, depending on the widths of wrapped fitting paths.

Definition 13.28. *Assume fitting paths $P_i = ((s_i, l, m_{i,j}, n_{i,j}, k_{i,j}))_{j \in [\![\theta_i]\!]}$ for $i \in [\![2]\!]$.*

1. *We say that P_1 is finer than or equivalent to P_2 if there is an order-preserving map $f : ([\![\theta_1]\!], <) \to ([\![\theta_2]\!], <)$ such that each $i \in [\![\theta_1]\!]$ satisfies*

$$m_{1,i} = m_{2,f(i)},$$
$$n_{1,i} = n_{2,f(i)}.$$

   When P_1 is finer than or equivalent to P_2, we write

$$P_1 \vdash P_2.$$

2. *We say that P_1 is equivalent to P_2 if θ_1 and θ_2 have the same widths and each $j \in [\![0, \theta_{1,2} - \theta_{1,1}]\!]$ satisfies*

$$m_{1,\theta_{1,1}+j} = m_{2,\theta_{2,1}+j},$$
$$n_{1,\theta_{1,1}+j} = n_{2,\theta_{2,1}+j}.$$

   When P_1 is equivalent to P_2, we write

$$P_1 \equiv P_2.$$

We prove the antisymmetricity of $\vdash$ on the equivalence relation $\equiv$ by the following binary relation.

Definition 13.29. *Suppose $m, m' \in \mathbb{Z}^l$ for $l \in \mathbb{Z}_{\geq 1}$. If $m < m'$ or $m^\vee < m'$, then let*

$$m \lessdot m'.$$

Lemma 13.30. *The binary relation $\lessdot$ is a strict partial order on $\mathbb{Z}^l$.*

Proof. First, we prove the irreflexivity. If $m \in \mathbb{Z}^l$, then $m < m$ does not hold. Also, $m < m^\vee$ does not hold either, since $m < m^\vee$ implies $m_1 < m_1^\vee = m_l < m_l^\vee = m_1$.

Second, we prove the transitivity. Let $m_1, m_2, m_3 \in \mathbb{Z}^l$ such that $m_1 \lessdot m_2 \lessdot m_3$. Then, $m_1 < m_2$ or $m_1 < m_2^\vee$, and $m_2 < m_3$ or $m_2 < m_3^\vee$. We have $m_1 < m_3$ or $m_1 < m_3^\vee$, since taking $^\vee$ preserves the order $<$. $\square$

We state the antisymmetricity of $\vdash$ with respect to the equivalence relation $\equiv$.

Proposition 13.31. *Suppose fitting paths $P_i = ((s_i, l, m_{i,j}, n_{i,j}, k_{i,j}))_{j \in [\![\theta_i]\!]}$ for $i \in [\![2]\!]$. If $P_1 \vdash P_2$ and $P_2 \vdash P_1$, then $P_1 \equiv P_2$. In particular, $\vdash$ is a partial order on fitting paths with respect to the equivalence relation $\equiv$.*

Proof. By $P_1 \vdash P_2$ and $P_2 \vdash P_1$, we have order-preserving maps $f : (\theta_1, <) \to (\theta_2, <)$ and $g : (\theta_2, <) \to (\theta_1, <)$ such that

$$m_{1,i} = m_{2,f(i)} \text{ for each } i \in [\![\theta_1]\!], \tag{13.6.1}$$
$$m_{2,i} = m_{1,g(i)} \text{ for each } i \in [\![\theta_2]\!]. \tag{13.6.2}$$

First, we prove

$$f(i+1) - f(i) = 1 \tag{13.6.3}$$

for each $i \in [\![\theta_1]\!]$. Suppose

$$f(i) < u < f(i+1) \tag{13.6.4}$$

for some $u \in [\![\theta_2]\!]$. Then,

$$m_{2,u} \lessdot m_{2,f(i+1)} = m_{1,i+1} \tag{13.6.5}$$

by Lemma 12.27 and equation (13.6.1). Let us show $g(u) < i+1$. If $g(u) > i+1$, then

$$m_{1,i+1} \lessdot m_{1,g(u)} = m_{2,u} \tag{13.6.6}$$

by Lemma 12.27 and equation (13.6.2). If $g(u) = i+1$, then

$$m_{1,i+1} = m_{1,g(u)} = m_{2,u} \tag{13.6.7}$$

by equation (13.6.2). By inequality (13.6.5), inequality (13.6.6) and equation (13.6.7) contradict Lemma 13.30.

Also, inequality (13.6.4) implies

$$m_{1,i} = m_{2,f(i)} \lessdot m_{2,u}.$$

As above, $i \geq g(u)$ implies $m_{2,u} = m_{1,g(u)} \lessdot m_{1,i}$ or $m_{2,u} = m_{1,g(u)} = m_{1,i}$. In either case, we have a contradiction to Lemma 13.30. It follows that $i < g(u) < i+1$ against $g(u) \in \mathbb{Z}$. Therefore, equation (13.6.3) follows.

Second, assume $f(\theta_{1,1}) > \theta_{2,1}$. Then, Lemmas 12.27 and 13.30 imply $g(\theta_{2,1}) < \theta_{1,1}$, which can not happen. It follows that $f(\theta_{1,1}) = \theta_{2,1}$. We deduce $P_1 \equiv P_2$ by equation (13.6.3).

We obtain the latter statement by the reflexivity and transitivity of $\vdash$. ☐

We introduce the notion of restricted fitting paths.

Definition 13.32. *Suppose a fitting path $P = (P_i)_{i \in [\![\theta_1]\!]}$. For a gate θ_2 such that $\theta_{1,1} \leq \theta_{2,1} \leq \theta_{2,2} \leq \theta_{1,2}$, we define the restricted fitting path*

$$r_{\theta_1,\theta_2}(P) = (P_i)_{i \in [\![\theta_2]\!]}.$$

Then, we realize width-one fitting paths by $\pi(\lambda, h)$ and the equivalence relation $\equiv$.

Lemma 13.33. *For $l = 1$, consider a fitting path $P = ((s, l, m_i, n_i, k_i))_{i \in [\![\theta_1]\!]}$. Then, we have*

$$P \equiv r_{\theta_1,\theta_2}(\pi(\lambda, h))$$

for some $(\lambda) = \sigma(k_{\theta_{1,1}}) \in \mathbb{Z}^l$, $(h) = m_{\theta_{1,1}} - \sigma(k_{\theta_{1,1}}) \in \mathbb{Z}^l$, and $\theta_2 = (1, \theta_{1,2} - \theta_{1,1} + 1) \in \hat{\mathbb{Z}}^2$.

Proof. The equivalent supports k_i give $\sigma(k_i) = (\lambda)$ for some $\lambda \in \mathbb{Z}_{\geq 1}$. Also, $m_i = n_i$ by $l = 1$. The assertion follows, since $m_i = n_i = (\lambda(i - \theta_{1,1} + 1) + h) \in \mathbb{Z}^l$ for some $h \in \mathbb{Z}$ such that $m_{\theta_{1,1}} = n_{\theta_{1,1}} = (h + \lambda)$. ☐

Furthermore, we obtain $\pi(1)$ as the finest fitting path among all the width-one wrapped fitting paths.

Proposition 13.34. *We have the following.*

1. *If P is a width-one wrapped fitting path, then $\pi(1) \vdash P$.*
2. *If there exists a wrapped fitting path P' such that $P' \vdash P$ for each width-one wrapped fitting path P, then $\pi(1) \equiv P'$.*

Proof. Proof of Claim 1. For some gates θ_1 and θ_2, $P \equiv r_{\theta_1,\theta_2}(\pi(\lambda,h))$ by Lemma 13.33. We obtain Claim 1, since $h \in \mathbb{Z}_{\geq 0}$ for the wrapped P.

*Proof of Claim 2.*Claim 2 holds by Proposition 13.31 and Claim 1. □

Next, for each $l \in \mathbb{Z}_{\geq 1}$, we prove the non-existence of finest fitting paths among all the width-l wrapped fitting paths. We state the following lemma.

Lemma 13.35. *Assume fitting paths $P = ((s,l,m_i,n_i,k_i))_{i \in [\![\theta]\!]}$. Let $u_1, u_2 \in [\![\theta]\!]$ such that $u_1 < u_2$. Then, there is $\mu \in \mathbb{Z}^2_{\geq 0}$ such that $\mu_1 \geq 1$, $\sum \mu = u_2 - u_1$, and*

$$\mu_1 \sigma(k_{u_1}) + \mu_2 \sigma(k_{u_1})^\vee + n^\vee_{u_1} = \begin{cases} m_{u_2}, \text{ or} & (13.6.8) \\ m^\vee_{u_2}. & (13.6.9) \end{cases}$$

Proof. Since $(n_{u_2-1} \boxplus k_{u_2-1})^\vee = m_{u_2}$, we have

$$\sigma(k_{u_2-1}) + n^\vee_{u_2-1} = m_{u_2}. \tag{13.6.10}$$

This gives the assertion when $u_2 - u_1 = 1$.

We demonstrate the assertion by the induction. Let $u_2 - u_1 \geq 2$. Also, let $\lambda \in \mathbb{Z}^2_{\geq 0}$ such that $\lambda_1 \geq 1$, $\sum \lambda = u_2 - u_1 - 1$, and

$$\lambda_1 \sigma(k_{u_1}) + \lambda_2 \sigma(k_{u_1})^\vee + n^\vee_{u_1} = \begin{cases} m_{u_2-1}, \text{ or} & (13.6.11) \\ m^\vee_{u_2-1}. & (13.6.12) \end{cases}$$

First, suppose equation (13.6.11). If $n^\vee_{u_2-1} = m_{u_2-1}$, then we obtain equation (13.6.8), adding equations (13.6.10) and (13.6.11). By equation (13.6.10), we deduce

$$\sigma(k_{u_2-1})^\vee + n_{u_2-1} = m^\vee_{u_2}. \tag{13.6.13}$$

Hence, $n_{u_2-1} = m_{u_2-1}$ gives equation (13.6.9) by equations (13.6.11) and (13.6.13).

Second, suppose equation (13.6.12). If $n_{u_2-1} = m^\vee_{u_2-1}$, then equation (13.6.9) follows from equations (13.6.12) and (13.6.13). Also, $n^\vee_{u_2-1} = m^\vee_{u_2-1}$ gives equation (13.6.8) by equations (13.6.10) and (13.6.12). □

We obtain the non-existence of finest fitting paths for higher-width wrapped fitting paths.

Proposition 13.36. *Let $l \geq 2$. Then, there is no width-l fitting path P such that $P \vdash P'$ for each width-l wrapped fitting path P'.*

Proof. We prove the assertion by contradiction. Suppose a fitting path $P = ((s,l,m_i,n_i,k_i))_{i \in [\![\theta]\!]}$ such that $P \vdash P'$ for each width-l wrapped fitting path P'.

Let $\xi = (l) \in Y(1,l)$, $s' = (0,\infty)$, and $\theta' = (1,\infty)$. Then, by Theorem 12.32 and Lemma 12.36, there is a fitting path $P' = \mathbb{P}_{s,l,\xi,0} = ((s',l,m'_i,n'_i,k'_i))_{i \in [\![\theta']\!]}$ such that

$$m'_i = n'_i = i\iota^l(1).$$

Furthermore, $P \vdash P'$ implies $m'_1 = m_{i_1}$ and $m'_2 = m_{i_2}$ for some $i_1, i_2 \in [\![\theta]\!]$ such that $i_1 < i_2$. Then, Proposition 13.35 gives $\mu \in \mathbb{Z}^2_{\geq 0}$ such that $\sum \mu = i_2 - i_1$, $\mu_1 \geq 1$, and

$$\mu_1 \sigma(k_1) + \mu_2 \sigma(k_1)^\vee = m_{i_2} - m_{i_1} = m'_2 - m'_1 = \iota^l(1).$$

Hence, $\sigma(k_1) > 0$ implies $i_2 - i_1 = 1$ and $\sigma(k_1) = \iota^l(1)$. Also, we obtain flat $m_j = n_j = (j - i_1 + 1)\iota^l(1)$ for $j \in [\![\theta]\!]$.

However, by Theorem 12.32 and Lemma 12.36, $\xi = (l,1) \in Y(2,l)$ gives a wrapped fitting path $P' = \mathbb{P}_{s,l,\xi,3} = ((s',l,m'_i,n'_i,k'_i))_{i \in [\![\theta']\!]}$ such that each $i \in [\![\theta']\!]$ satisfies

$$m'_i = n'_i = (5) + \iota^{l-1}(4) + (i-1)\iota^l(3).$$

Then, $P \vdash P'$ can not hold, since each m_i of P is flat. $\square$

By Propositions 13.34 and 13.36, the finest fitting path exists only for width-one wrapped fitting paths. We simply call $\pi(1)$ the finest fitting path for our convenience.

13.7 On phase transitions of width-one monomial parcels

We obtain some phase transitions of width-one monomial parcels by canonical mediators and the finest fitting path $\pi(1)$. This uses the following lemma on tame factors.

Lemma 13.37. *Let $l = 1$ and $\mu = (l,w,\gamma)$ be a monomial index. Consider the t-monomials $\Psi_{s,\gamma,q}$. Assume $r \in \mathrm{O}_{\mathfrak{X}}$ such that $0 < q(r) < 1$. Then, for $i \in \mathbb{Z}_{\geq 0}$, we have*

$$\lim_{i\to\infty} \frac{\Psi_{s,\gamma,q(r),(i+1)}}{\Psi_{s,\gamma,q(r),(i)}} = \begin{cases} q(r)^{\gamma_{1,2}} & \text{if } \gamma_{1,1} = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. The monomial conditions of μ yield $\gamma_{1,1} \geq 0$. Hence, the assertion holds by

$$\lim_{i\to\infty} \frac{\Psi_{s,\gamma,q(r),(i+1)}}{\Psi_{s,\gamma,q(r),(i)}} = \lim_{i\to\infty} \frac{q(r)^{\gamma_{1,1}(i+1)^2+\gamma_{1,2}(i+1)}}{q(r)^{\gamma_{1,1}i^2+\gamma_{1,2}i}} = q(r)^{\gamma_{1,2}} \lim_{i\to\infty} q(r)^{(2i+1)\gamma_{1,1}}.$$

$\square$

We then state the phase transitions.

Proposition 13.38. *Suppose a monomial parcel $\mathscr{F} = \Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\phi,x,\mathfrak{X})$ such that $s = (0,\infty)$, $l = 1$, and $\phi(x) = (1-q) \in \mathbb{Q}(\mathfrak{X})^l$. Then, for the merged pair $\zeta = \Omega(\mathscr{F})$, we have the following.*

1. *ζ has no rear critical points.*
2. *ζ has no asymptotic critical points.*
3. *If q is fully admissible by $\mathfrak{X}$, then the following statements are equivalent:*

(a) ζ *has a front semi-phase transition;*

(b) $\gamma_{1,1}+\gamma_{1,2}>0$*;*

(c) ζ *has a front phase transition.*

Moreover, if $\mathfrak{X}=\{X_1\}$ *and* $\gamma_{1,1}+\gamma_{1,2}>0$*, then* ζ *has a single phase transition at the front critical point.*

Proof. Proof of Claim 1. Claim 1 holds, since $\pi(1)$ is infinite-length.

Proof of Claim 2. Let $r\in \mathrm{O}_{\mathfrak{X}}$. Then, Claim 1 of Lemma 5.22 implies $0<q(r)<1$. Hence, by Lemma 13.37, $\mathscr{F}$ is tame along P by $t\in\mathbb{Q}(\mathfrak{X})$ such that $t(r)=q(r)^{\gamma_{1,2}}$ if $\gamma_{1,1}=0$, and 0 otherwise. Let $\pi(1)=((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$. Then, $\sigma(k_{\theta_1})=(1)\in\mathbb{Z}^l$ yields

$$\mathrm{A_D}(\zeta)=\prod\phi(x)^{\sigma(k_{\theta_1})^{\vee}\circ w}-t\cdot\prod\phi(x)^{\sigma(k_{\theta_1})^{\vee}\circ w}=(1-q)^{w_1}(1-t).$$

Thus, $\mathrm{A_D}(\zeta)=0$ has no solutions over $\mathrm{O}_{\mathfrak{X}}$.

Proof of Claim 3. Claim 3 follows from Claim 2 of Proposition 13.26, since

$$t_\gamma(m_{\theta_1})-t_\gamma(m_{\theta_1}\boxminus k_{\theta_1})=\gamma_{1,1}+\gamma_{1,2}.$$

Then, Claims 1 and 2 give the latter statement of Claim 3. □

14 Explicit critical points, phase transitions, and merged determinants

We adopt the following notation for simplicity.

Definition 14.1. *Consider a* $(\theta,\mathfrak{X})$*-merged pair* $\zeta=(P,\mathscr{F})$ *such that* $\mathfrak{X}=\{X_1\}$*. Suppose* $(h)\in\mathrm{O}_{\mathfrak{X}}$ *for a real number* $0<h<1$.

1. *For a path-parcel sequence, let*

$$u(\zeta,h)=u(\zeta,(h)).$$

2. *We call h a front critical point of* ζ*, rear critical point of* ζ*, or asymptotic critical point of* ζ *if* (h) *is a*
   - *front critical point of* ζ*,*
   - *rear critical point of* ζ*, or*
   - *asymptotic critical point of* ζ*, respectively.*

We recall the following parcel $\mathscr{L}$, which appears in equation (1.4.3) by a different terminology in the introduction.

Definition 14.2. *Let* $s=(0,\infty)$*,* $l=1$*,* $w=(1)$*, and* $\mathfrak{X}=\left\{q^{\frac{1}{2}}\right\}$*. Then, we define the* $q^{\frac{1}{2}}$*-linear monomial parcel*

$$\mathscr{L}=\Lambda(s,l,w,\succ,\Psi_{s,\left(\left(0,\frac{1}{2},0\right)\right),q},x,\mathfrak{X}).$$

14.1 Golden angle as a critical point

For $l = 1$, $w = (2)$, $s = (0, \infty)$, and $\mathfrak{X} = \{q\}$, let

$$\mathscr{F} = \Lambda(s, l, w, >_q, \Psi_{s,((0,1,0)),q}, x, \mathfrak{X})$$

such that $\mathscr{F}_\lambda = \frac{q^{\lambda_1}}{(\lambda_1)_q^2}$ for $\lambda \in [\![s]\!]^l$. Also, by Theorem 8.40 and Proposition 13.26, we consider the ideal merged pair $\zeta = \Omega(\mathscr{F})$ for a critical point and phase transition, since the path-parcel sequences $u(\zeta, r)$ of $r \in \mathrm{O}_{\mathfrak{X}}$ are almost strictly unimodal by Theorem 12.18.

14.1.1 On critical points and phase transitions

Let $\pi(1) = ((s, l, m_i, n_i, k_i))_{i \in [\![\theta]\!]}$ with $m_0 = n_0 = (0)$ and $k_0 = a_0 = b_0 = (0, 1)$. Then, a real number $0 < q < 1$ is a front critical point of ζ if and only if

$$\mathscr{F}_{m_0}(q) = 1 = \frac{q}{(1-q)^2} = \mathscr{F}_{m_1}(q).$$

Therefore, the unique front critical point $\mathrm{F_C}(\zeta)$ of ζ is the golden angle: i.e.,

$$\mathrm{F_C}(\zeta) = \frac{3 - \sqrt{5}}{2} = 0.381966\ldots.$$

By Proposition 13.38, ζ has neither rear nor asymptotic critical points. However, ζ has the front phase transition at $\mathrm{F_C}(\zeta)$ such that

- $u(\zeta, 0.3)$ is a strictly decreasing sequence,
- $u(\zeta, \mathrm{F_C}(\zeta))$ is a decreasing and hill sequence, and
- $u(\zeta, 0.4)$ is a two-sided hill.

For each $i \in [\![0, 5]\!]$, Figure 6 plots the bottom point for $\mathscr{F}_{m_i}(0.3)$, the middle point for $\mathscr{F}_{m_i}(\mathrm{F_C}(\zeta))$, and the top point for $\mathscr{F}_{m_i}(0.4)$.

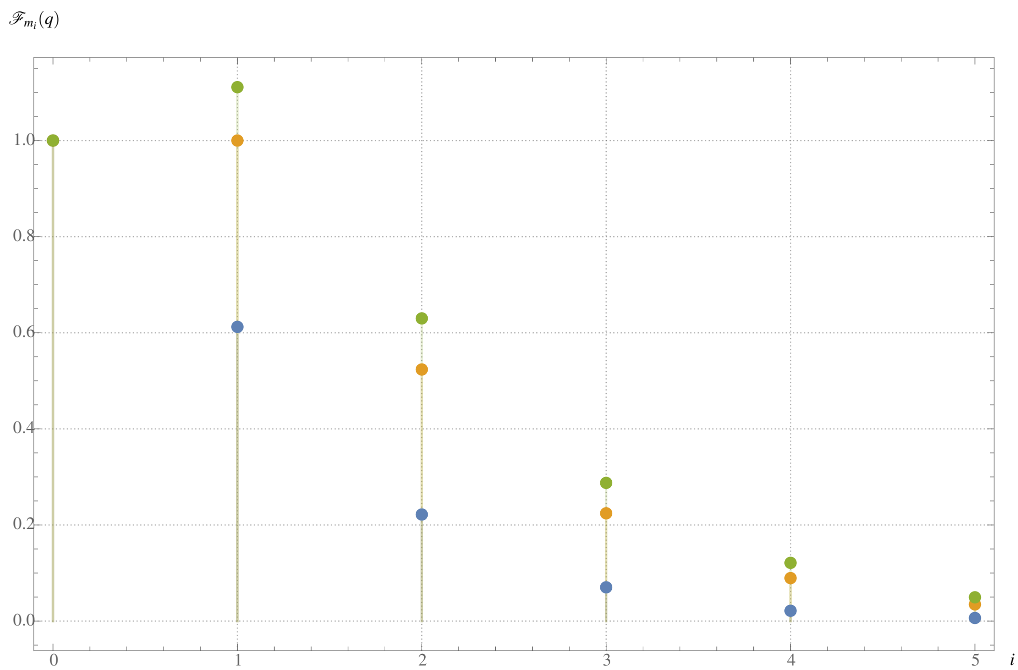


Figure 6: $\mathscr{F}_{m_i}(q)$ of $q = 0.3$ (bottom), $\mathrm{F_C}(\zeta)$ (middle), and 0.4 (top)

14.1.2 Polynomials with positive integer coefficients of an ideal merged pair

For $i \in \mathbb{Z}_{\geq 1}$, ζ gives the following q-polynomials with positive coefficients:

$$\begin{aligned}\Delta(\mathscr{F})(s,l,w,m_i,n_i,k_i,x,\mathfrak{X}) &= \frac{(i)_q^2 \cdot (i+1)_q^2}{(0)_q^2 \cdot (1)_q^2} \cdot (\mathscr{F}_{1,m_i}\mathscr{F}_{1,n_i} - \mathscr{F}_{1,m_{i-1}}\mathscr{F}_{1,n_{i+1}}) \\ &= \frac{(i)_q^2 \cdot (i+1)_q^2}{(0)_q^2 \cdot (1)_q^2} \cdot \left(\left(\frac{q^i}{(i)_q^2} \right)^2 - \frac{q^{i-1}}{(i-1)_q^2} \frac{q^{i+1}}{(i+1)_q^2} \right).\end{aligned}$$

Explicitly, some of them are

$$\begin{aligned}\Delta(\mathscr{F})(s,l,w,m_1,n_1,k_1,x,\mathfrak{X}) &= 2q^3 + q^4, \\ \Delta(\mathscr{F})(s,l,w,m_2,n_2,k_2,x,\mathfrak{X}) &= 2q^6 + 2q^7 + q^8, \\ \Delta(\mathscr{F})(s,l,w,m_3,n_3,k_3,x,\mathfrak{X}) &= 2q^9 + 2q^{10} + 2q^{11} + q^{12}, \\ \Delta(\mathscr{F})(s,l,w,m_4,n_4,k_4,x,\mathfrak{X}) &= 2q^{12} + 2q^{13} + 2q^{14} + 2q^{15} + q^{16}, \\ \Delta(\mathscr{F})(s,l,w,m_5,n_5,k_5,x,\mathfrak{X}) &= 2q^{15} + 2q^{16} + 2q^{17} + 2q^{18} + 2q^{19} + q^{20}.\end{aligned}$$

14.1.3 Golden angle from golden ratio as critical points

The $q^{\frac{1}{2}}$-linear $\mathscr{L}$ has the golden ratio as the front critical point in Section 1.7. Moreover, $\mathscr{L}$ gives $\mathscr{F}$, because

$$\mathscr{L}^{\circ 2} = \Lambda(s,l,w,\succ,\Psi_{s,((0,1,0)),q},x,\left\{q^{\frac{1}{2}}\right\}),$$

and $\mathscr{F} = r_{(1),(2)}(\mathscr{L}^{\circ 2})$ by the parcel restriction $r_{(1),(2)}$ (or $\mathscr{F} = \mathscr{L}^{\circ 2}$ in $\mathbb{Q}(q^{\frac{1}{2}})$).

14.2 A non-canonical mediator with phase transitions

Consider $s = (0, \infty)$, $l = 1$, $w = (2)$, $\rho = (1)$, and $\mathfrak{X} = \{q\}$. Also, let

$$\phi(x) = \left(\frac{29}{20} - 5q + 5q^2\right) \in \mathbb{Q}(x)^l.$$

Then, $B(s,l,w,m,\phi,\rho,x,\mathfrak{X}) >_q 0$ for each $m \in [\![s]\!]^l$ by $\rho = (1)$ and Claim 1 of Lemma 4.2. Hence, ϕ is a non-canonical $(s,l,w,>_q,\rho,x,\mathfrak{X})$-mediator, because $\phi_1(x_1)^{w_1} >_{\mathrm{O}_{\mathfrak{X}}} 0$ by

$$\frac{29}{20} - 5q + 5q^2 = 5\left(q - \frac{1}{2}\right)^2 + \frac{1}{5}. \tag{14.2.1}$$

Now, these analogs $\phi_1(x_1)^{\lambda_1}[\lambda_1]!_q$ of q-Pochhammer symbols $(\lambda_1)_q$ give the monomial parcel

$$\mathscr{F} = \Lambda(s,l,w,>_q,\Psi_{s,((0,1,0)),q},\phi,\rho,x,\mathfrak{X})$$

such that each $\lambda \in [\![s]\!]^l$ satisfies

$$\mathscr{F}_\lambda = \frac{q^{\lambda_1}}{\left(\frac{29}{20} - 5q + 5q^2\right)^{2\lambda_1} \cdot [\lambda_1]!_q^2}.$$

By Theorem 8.40, $\mathscr{F}$ is $>_q$-merged-log-concave. Hence, we have the ideal merged pair $\zeta = \Omega(\mathscr{F})$, which gives the same merged determinants in Section 14.1 by Proposition 7.2.

14.2.1 On critical points and phase transitions

Let $P = \pi(1) = ((s,l,m_i,n_i,k_i))_{i \in [\![\theta]\!]}$ with $m_0 = n_0 = (0)$ and $k_0 = a_0 = b_0 = (0,1)$. First, unlike the canonical mediator in Section 14.1, the non-canonical ϕ of ζ gives exactly the two front critical points by equation (14.2.1). More explicitly, we have the front critical points $\mathrm{F_C}(\zeta)_1 = 0.253594\ldots$ and $\mathrm{F_C}(\zeta)_2 = 0.884885\ldots$ that solve

$$\mathscr{F}_{m_0}(q) = 1 = \frac{q}{\left(\frac{29}{20} - 5q + 5q^2\right)^2} = \mathscr{F}_{m_1}(q).$$

Second, there are no rear critical points for the infinite-length P. Third, $\mathscr{F}$ is tame along P by q in Lemma 13.37. Thus, an asymptotic critical point $0 < q < 1$ solves

$$\mathrm{A_D}(\zeta) = \left(\frac{29}{20} - 5q + 5q^2\right)^2 - q(1-q)^2 = 0.$$

For $v(q) = q(1-q)^2$, $v'(q) = (1-q)(1-3q)$ and $v\left(\frac{1}{3}\right) > \phi_1\left(\frac{1}{3}\right)^2$. Then, equation (14.2.1) gives the two asymptotic critical points $\mathrm{A_C}(\zeta)_1 = 0.30813\ldots$ and $\mathrm{A_C}(\zeta)_2 = 0.63486\ldots$. Hence, we obtain the following table of phase transitions.

	0		$\mathrm{F_C}(\zeta)_1$		$\mathrm{A_C}(\zeta)_1$		$\mathrm{A_C}(\zeta)_2$		$\mathrm{F_C}(\zeta)_2$		1
$\mathrm{F_D}(\zeta)$	$+$	$+$	0	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$
$\mathrm{A_D}(\zeta)$	$+$	$+$	$+$	$+$	0	$-$	0	$+$	$+$	$+$	$+$

In particular, the non-canonical mediator gives not only the front phase transitions at $\mathrm{F_C}(\zeta)_1$ and $\mathrm{F_C}(\zeta)_2$, but also the asymptotic phase transitions at $\mathrm{A_C}(\zeta)_1$ and $\mathrm{A_C}(\zeta)_2$ between the front phase transitions.

First, the front phase transition at $\mathrm{F_C}(\zeta)_1$ gives the strictly decreasing $u(\zeta,0.2)$, decreasing and hill $u(\zeta,\mathrm{F_C}(\zeta)_1)$, and two-sided hill $u(\zeta,0.3)$. For each $i \in [\![0,5]\!]$, Figure 7 plots $\mathscr{F}_{m_i}(q)$ of $q=0.2$ for the bottom point, $q=\mathrm{F_C}(\zeta)_1$ for the middle point, and $q=0.3$ for the top point.

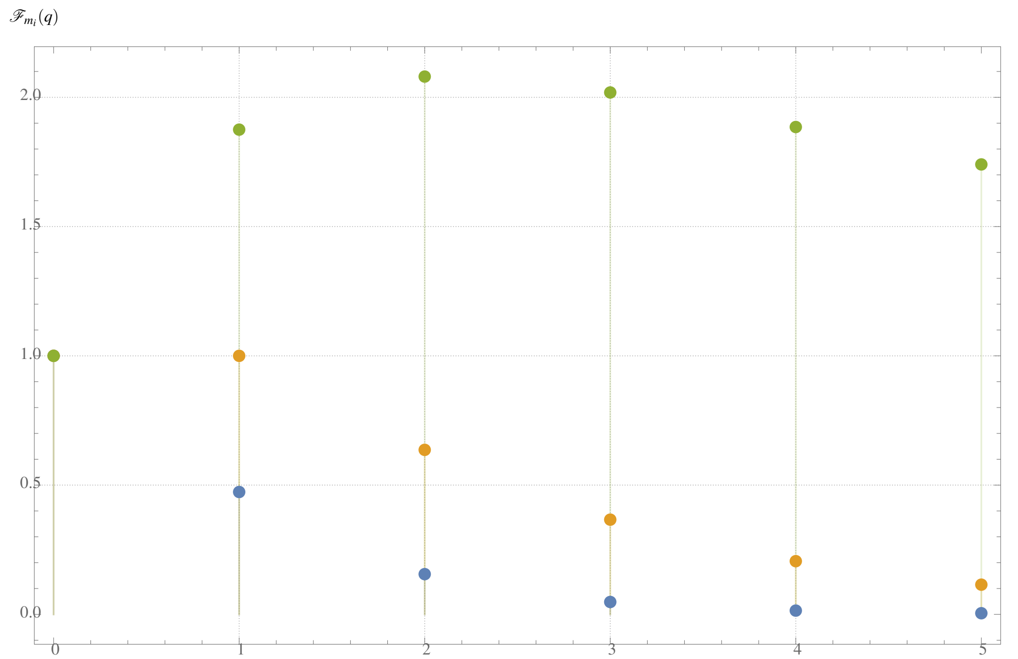


Figure 7: $\mathscr{F}_{m_i}(q)$ of $q=0.2$ (bottom), $\mathrm{F_C}(\zeta)_1$ (middle), and 0.3 (top)

Second, the front phase transition at $\mathrm{F_C}(\zeta)_1$ gives the two-sided hill $u(\zeta,0.8)$, decreasing and hill $u(\zeta,\mathrm{F_C}(\zeta)_2)$, and strictly decreasing $u(\zeta,0.95)$. Figure 8 puts $\mathscr{F}_{m_i}(q)$ of $q=0.95$ for the bottom point, $q=\mathrm{F_C}(\zeta)_2$ for the middle point, and $q=0.8$ for the top point.

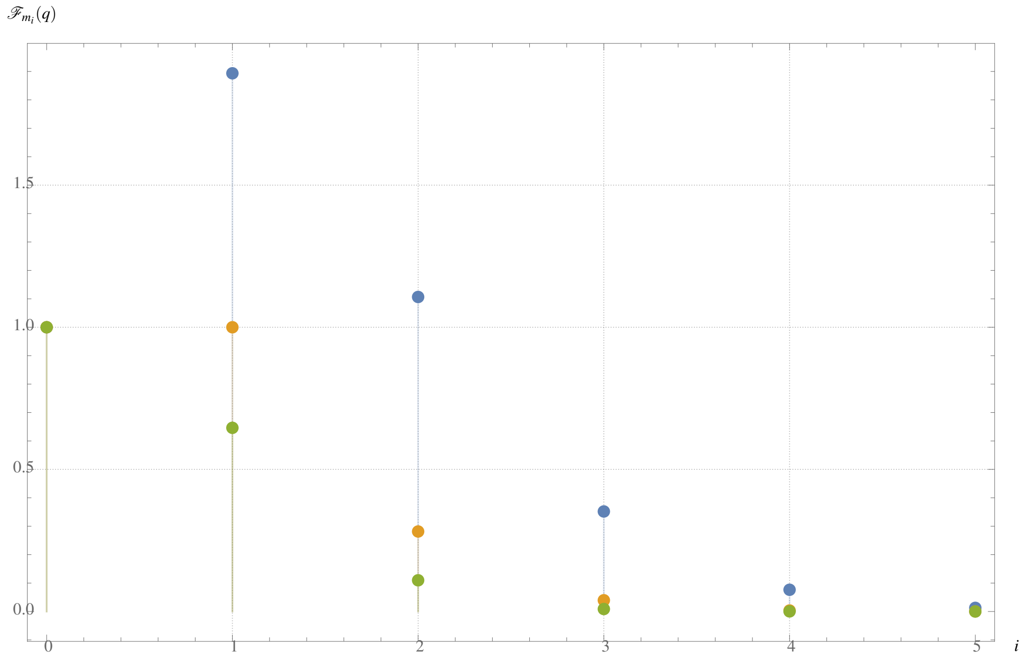


Figure 8: $\mathscr{F}_{m_i}(q)$ of $q = 0.95$ (bottom), $\mathrm{F_C}(\zeta)_2$ (middle), and 0.8 (top)

Third, Proposition 13.7 gives the asymptotic hills $u(\zeta, \mathrm{A_C}(\zeta)_1)$ and $u(\zeta, \mathrm{A_C}(\zeta)_2)$. Also, we obtain the strictly increasing sequence $u(\zeta, q)$ for $\mathrm{A_C}(\zeta)_1 < q = 0.5 < \mathrm{A_C}(\zeta)_2$. Figure 9 plots $\log(\mathscr{F}_{m_i}(q))$ of $q = \mathrm{A_C}(\zeta)_1$ for the bottom point, $q = \mathrm{A_C}(\zeta)_2$ for the middle point, and $q = 0.5$ for the top point.

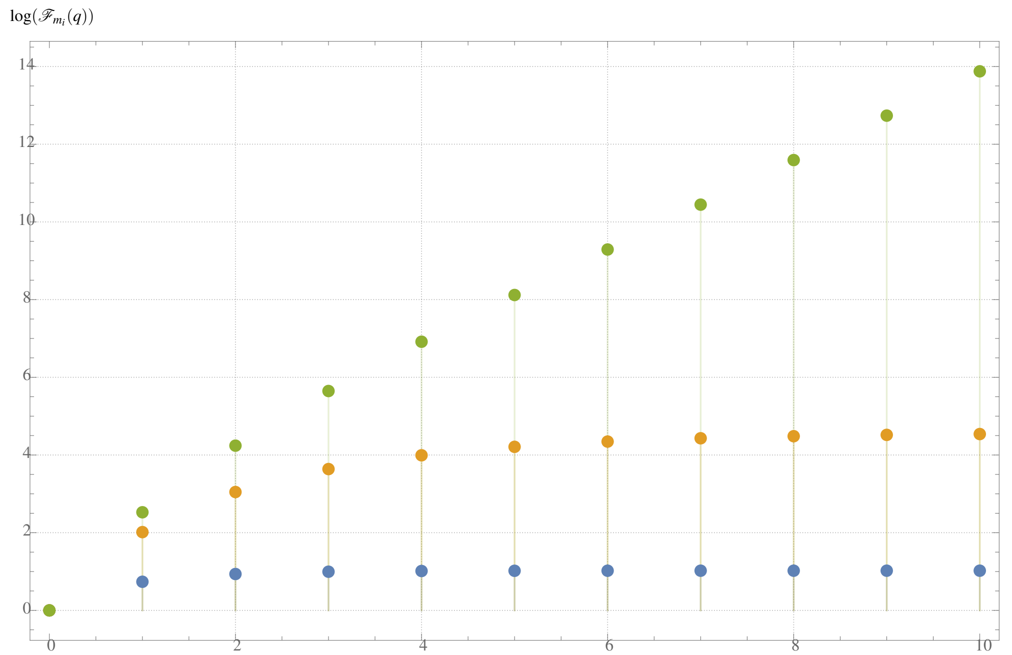


Figure 9: $\log(\mathscr{F}_{m_i}(q))$ of $q = \mathrm{A_C}(\zeta)_1$ (bottom), $\mathrm{A_C}(\zeta)_2$ (middle), and 0.5 (top)

14.3 A non-canonical mediator only with a semi-phase transition

We have a parcel $\mathscr{F}$ with a front semi-phase transition, but without a front phase transition. Consider $s = (0, \infty)$, $l = 1$, $w = (1)$, $\rho = (1)$, and $\mathfrak{X} = \{q\}$. Also, let

$$\phi(x) = \left(\frac{3q^2}{2} - \frac{q}{2} + \frac{3}{8}\right) \in \mathbb{Q}(x)^l.$$

Then, ϕ is a $(s, l, w, >_q, \rho, x, \mathfrak{X})$-mediator as in Section 14.2, since $\frac{3q^2}{2} - \frac{q}{2} + \frac{1}{8} = \frac{3}{2}\left(q - \frac{1}{6}\right)^2 + \frac{1}{3}$ and $\rho = (1)$. Hence, the analogs $\phi_1(x_1)^{\lambda_1}[\lambda_1]!_q$ of q-Pochhammer symbols $(\lambda_1)_q$ give the $>_q$-merge-log-concave monomial parcel

$$\mathscr{F} = \Lambda(s, l, w, >_q, \Psi_{s,((0,1,0)),q}, \phi, \rho, x, \mathfrak{X})$$

such that $\mathscr{F}_\lambda = \frac{q^{\lambda_1}}{\phi_1(x_1)^{\lambda_1}[\lambda_1]!_q}$ for $\lambda \in [\![s]\!]^l$. This also gives the ideal merged pair $\zeta = \Omega(\mathscr{F})$.

14.3.1 On critical points and phase transitions

Let $P = \pi(1) = ((s, l, m_i, n_i, k_i))_{i \in [\![\theta]\!]}$ with $m_0 = n_0 = (0)$ and $k_0 = a_0 = b_0 = (0, 1)$. Then, ζ has the single front critical point $\mathrm{F_C}(\zeta) = \frac{1}{2}$ that solves

$$\mathscr{F}_{m_0}(q) = 1 = \frac{q}{\frac{3q^2}{2} - \frac{q}{2} + \frac{3}{8}} = \mathscr{F}_{m_1}(q).$$

As $\mathrm{F_Z}(\zeta) \neq \mathrm{O}_{\mathfrak{X}}$, ζ has the front semi-phase transition at $\mathrm{F_C}(\zeta)$. But, ζ has no front phase transitions, because $\mathscr{F}_{m_0}(q) \geq_{\mathrm{O}_{\mathfrak{X}}} \mathscr{F}_{m_1}(q)$ by $\frac{3q^2}{2} - \frac{q}{2} + \frac{3}{8} - q = \frac{3}{2}(q - \frac{1}{2})^2 \geq 0$.

There are no rear critical points for the infinite-length P. By Lemma 13.37, $\mathscr{F}$ is tame along P by q. But, there are no asymptotic critical points, since $\phi_1(q) >_{\mathrm{O}_{\mathfrak{X}}} \frac{1}{4}$ implies

$$\mathrm{A_D}(\zeta) = \left(\frac{3q^2}{2} - \frac{q}{2} + \frac{3}{8}\right) - q(1-q) >_{\mathrm{O}_{\mathfrak{X}}} 0.$$

14.3.2 Polynomials with positive integer coefficients of an ideal merged pair

The merged determinants of ζ give the following q-polynomials with positive coefficients:

$$\Delta(\mathscr{F})(s,l,w,m_1,n_1,k_1,x,\mathfrak{X}) = q^3;$$
$$\Delta(\mathscr{F})(s,l,w,m_2,n_2,k_2,x,\mathfrak{X}) = q^6;$$
$$\Delta(\mathscr{F})(s,l,w,m_3,n_3,k_3,x,\mathfrak{X}) = q^9;$$
$$\dots.$$

14.4 A weight-zero parcel with critical points and without semi-phase transitions

For $s = (1,\infty)$, $l = 1$, and $\mathfrak{X} = \{q\}$, consider the weight-zero q-number parcel

$$\mathscr{F} = \Lambda(s,l,>_q,\chi_{s,l,q},\mathfrak{X}).$$

Also, there is the fitting path $P = \pi(\lambda,h) = ((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$ with $m_0 = n_0 = m_1 \boxminus k_1 = (h) \geq 1$. Then, we have the ideal merged pair $\zeta = (P,\mathscr{F})$.

First, ζ has no front critical points by $\mathscr{F}_{m_1} - \mathscr{F}_{m_0} >_q 0$. Second, ζ has no rear critical points either, since P is infinite-length. Third, for each $r \in \mathrm{O}_{\mathfrak{X}}$, $u(\zeta,r)$ is an asymptotic hill, since $0 < q < 1$ implies

$$\lim_{i\to\infty} \frac{\chi_{s,l,q,m_{i+1}}}{\chi_{s,l,q,m_i}} = \lim_{i\to\infty} \frac{1+q+\cdots+q^{m_{i+1,1}-1}}{1+q+\cdots+q^{m_{i,1}-1}} = \lim_{i\to\infty} \frac{1-q^{m_{i+1,1}}}{1-q^{m_{i,1}}} = 1.$$

Consequently, ζ has asymptotic critical points without semi-phase transitions.

14.5 A finite-length merged pair with a rear phase transition

For $s = (1,3)$, $l = 1$, and $\mathfrak{X} = \{q\}$, we have the $\geq_q$-merged-log-concave q-Stirling parcel $\mathscr{F} = \Lambda(s,l,>_q,S_{s,l,q},\mathfrak{X})$. Then, Claim 2 of Corollary 10.4 gives the finite-length $>_q$-merged-log-concave parcel

$$\mathscr{G} = \Lambda(s,l,w,>_q,S_{s,l,q},x,\mathfrak{X})$$

for $w = (1)$ and $x = (q)$. Moreover, $\theta = (1,1)$ gives the fitting path $P = r_{(1,\infty),\theta}(\pi(1,1)) = ((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$ such that $m_1 = n_1 = (2)$ and $k_1 = (0,1)$. Therefore, we obtain the finite-length ideal merged pair $\zeta = (P,\mathscr{G})$. Let $m_0 = n_0 = (1)$, $m_2 = n_2 = (3)$, and $k_0 = k_2 = k_1$ for our convenience.

14.5.1 On critical points and phase transitions

We have no front critical points of ζ, because

$$\mathscr{G}_{m_2}(q) = \frac{q+2}{(1-q)(1-q^2)} >_{\mathrm{O}_{\mathfrak{X}}} \frac{1}{1-q} = \mathscr{G}_{m_1}(q).$$

Since P is finite-length, ζ has no asymptotic critical points either. However, ζ has the rear critical point $\mathrm{R_C}(\zeta) = 0.86676\ldots$ that solves

$$\mathscr{G}_{m_2}(q) = \frac{q+2}{(1-q)(1-q^2)} = \frac{1}{(1-q)(1-q^2)(1-q^3)} = \mathscr{G}_{m_3}(q).$$

In particular, Theorem 13.19 gives the rear phase transition of ζ such that

- $u(\zeta, 0.84)$ is a two-sided hill,
- $u(\zeta, \mathrm{R_C}(\zeta))$ is a hill and increasing sequence, and
- $u(\zeta, 0.9)$ is a strictly increasing sequence.

In Figure 10, each $i \in [\![3]\!]$ puts the bottom point for $\mathscr{G}_{m_i}(0.84)$, the middle point for $\mathscr{G}_{m_i}(\mathrm{R_C}(\zeta))$, and the top point for $\mathscr{G}_{m_i}(0.9)$.

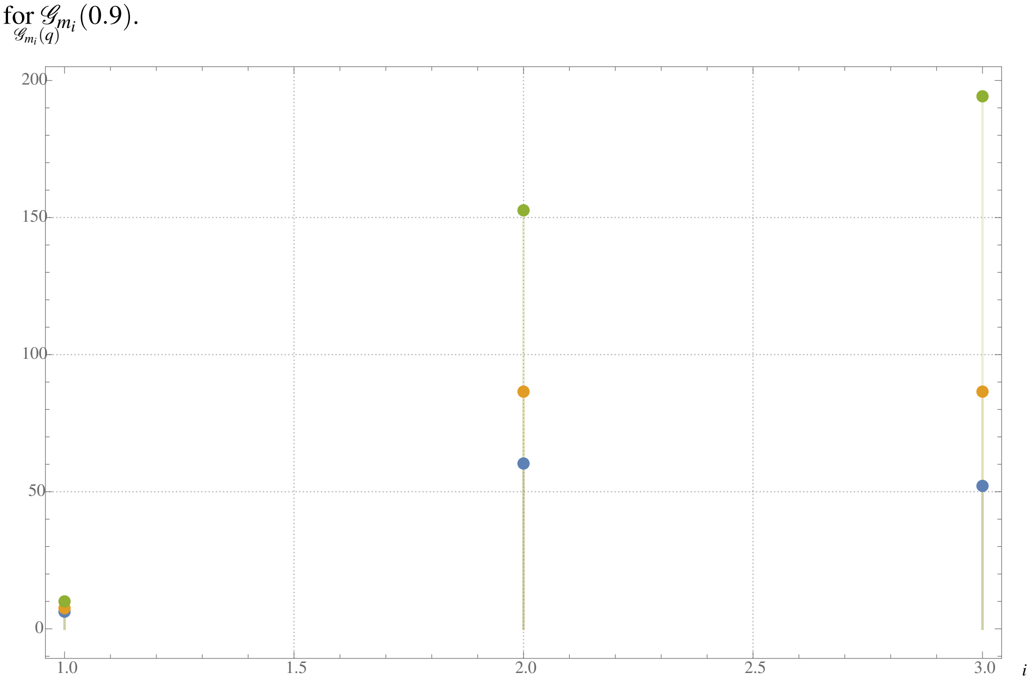


Figure 10: $\mathscr{G}_{m_i}(q)$ of $q = 0.84$ (bottom), $\mathrm{R_C}(\zeta)$ (middle), and 0.9 (top)

14.5.2 Polynomials with positive integer coefficients of an ideal merged pair

The ideal merged pair ζ gives the following q-polynomials with positive coefficients:

$$\begin{aligned}
\Delta(\mathscr{G})(s,l,w,m_0,n_0,k_0,x,\mathfrak{X}) &= \frac{(1)_q(2)_q}{(0)_q(1)_q}\cdot\left(\frac{1}{(1-q)}\right)^2 \\
&= 1+q; \\
\Delta(\mathscr{G})(s,l,w,m_1,n_1,k_1,x,\mathfrak{X}) &= \frac{(2)_q(3)_q}{(0)_q(1)_q}\cdot\left(\left(\frac{q+2}{(1-q)(1-q^2)}\right)^2\right. \\
&\quad \left. - \frac{1}{(1-q)}\cdot\frac{1}{(1-q)(1-q^2)(1-q^3)}\right) \\
&= 3+7q+9q^2+5q^3+q^4; \\
\Delta(\mathscr{G})(s,l,w,m_2,n_2,k_2,x,\mathfrak{X}) &= \frac{(3)_q(4)_q}{(0)_q(1)_q}\cdot\left(\frac{1}{(1-q)(1-q^2)(1-q^3)}\right)^2 \\
&= 1+q+q^2+q^3.
\end{aligned}$$

14.6 A higher-width parcel with a phase transition and conjectures

We discuss a higher-width parcel, unlike the examples above in Section 14. Let $l=3$, $s=(0,\infty)$, $w=(1)$, $\mathfrak{X}=\left\{q^{\frac{1}{2}}\right\}$, and $\succ=>_{q^{\frac{1}{2}}}$. Also, let $\gamma=\left(\left(\frac{1}{2},0,0\right),\left(-\frac{1}{2},0,0\right),\left(\frac{1}{2},0,0\right)\right)$. Then, we obtain the width-three monomial parcel $\mathscr{F}=\Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},x,\mathfrak{X})$ such that each $\lambda\in[\![s]\!]^l$ satisfies

$$\mathscr{F}_\lambda=\frac{q^{\frac{\lambda_1^2-\lambda_2^2+\lambda_3^2}{2}}}{(\lambda_1)_q(\lambda_2)_q(\lambda_3)_q}.$$

Since γ is palindromic, $\mathscr{F}$ is flip-invariant.

Moreover, let $\xi=(2,1)$, $h=4$, and $\theta=(1,\infty)$. Then, Theorem 12.32 gives the infinite-length fitting path $P=\mathbb{P}_{s,l,\xi,h}=((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$ such that each $i\in[\![\theta]\!]$ gives

$$\begin{aligned}
m_i=n_i&=(6,5,4)+(i-1)\cdot\iota^l(4), \\
a_i=\nu(k_i)&=(0,1,2,4,5,6), \\
b_i=\nu(m_i,n_i,k_i)&=(6,6,6,10,10,10)+(i-1)\cdot\iota^{2l}(4), \\
\sigma(k_i)&=(6,4,2).
\end{aligned}$$

Then, we have the merged pair $\zeta=(P,\mathscr{F})$. Let $m_0=n_0=(2,1,0)$, $k_0=k_1$, $a_0=\nu(k_0)$, and $b_0=\nu(m_0,n_0,k_0)$ for our convenience.

14.6.1 On critical points and phase transitions

As s is infinite, ζ has no rear critical points. Moreover, $\mathscr{F}$ is tame along P by 0, because $0<q^{\frac{1}{2}}<1$ gives

$$\lim_{i\to\infty}\frac{\Psi_{s,\gamma,q,m_{i+1}}}{\Psi_{s,\gamma,q,m_i}}=\lim_{i\to\infty}\frac{q^{\frac{(4(i+1)+2)^2-(4(i+1)+1)^2+(4(i+1))^2}{2}}}{q^{\frac{(4i+2)^2-(4i+1)^2+(4i)^2}{2}}}=\lim_{i\to\infty}q^{16i+12}=0.$$

Thus, ζ has no asymptotic critical points, since $u(\zeta, r)$ is a hill or decreasing sequence for each $r \in \mathrm{O}_{\mathfrak{X}}$ by Claim 1 of Proposition 13.12.

We have $t_\gamma(m_1) - t_\gamma(m_0) = 12 > 0$. Also, there is the front critical point $\mathrm{F_C}(\zeta) = 0.82439\ldots$ that solves

$$\mathscr{F}_{m_0}(q^{\frac{1}{2}}) = \frac{q^{\frac{3}{2}}}{(2)_q(1)_q(0)_q} = \frac{q^{\frac{27}{2}}}{(6)_q(5)_q(4)_q} = \mathscr{F}_{m_1}(q^{\frac{1}{2}}).$$

Claim 2 of Proposition 13.26 implies the unique front phase transition of ζ at $\mathrm{F_C}(\zeta)$. In particular,

- $u(\zeta, 0.8)$ is a strictly decreasing sequence,
- $u(\zeta, \mathrm{F_C}(\zeta))$ is a hill and decreasing sequence, and
- $u(\zeta, 0.83)$ is a two-sided hill.

For each $i \in [\![0,3]\!]$, Figure 11 gives the bottom point for $\log(\mathscr{F}_{m_i}(0.8))$, the middle point for $\log(\mathscr{F}_{m_i}(\mathrm{F_C}(\zeta)))$, and the top point for $\log(\mathscr{F}_{m_i}(0.83))$. Figure 11 takes the log scale to avoid point collisions.

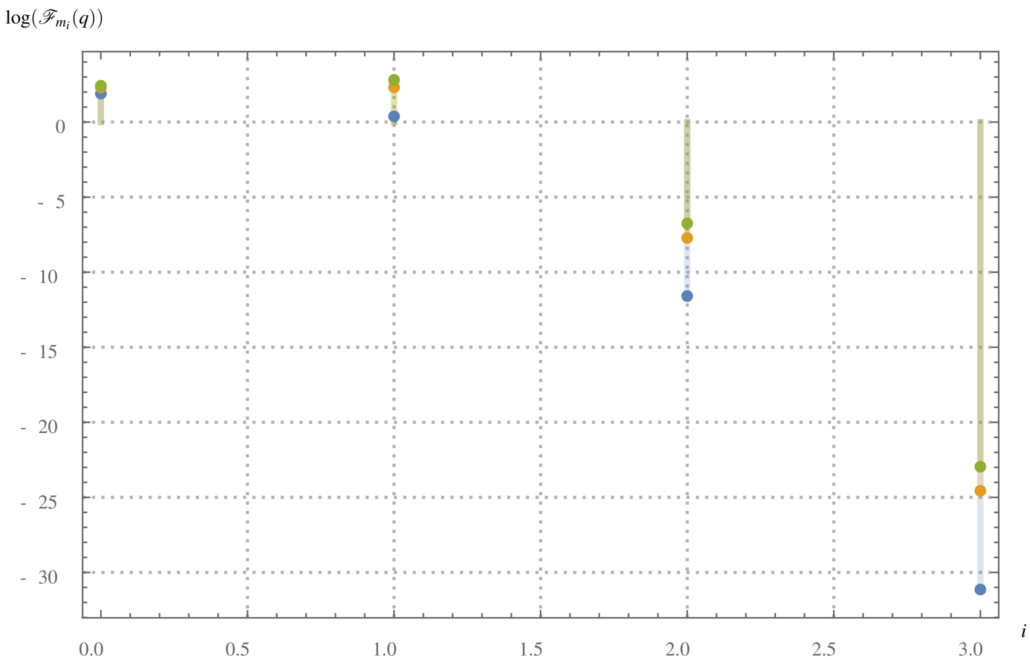


Figure 11: $\log(\mathscr{F}_{m_i}(q))$ of $q = 0.8$ (bottom), $\mathrm{F_C}(\zeta)$ (middle), and 0.83 (top)

14.6.2 Ideal property of a merged pair

The parcel $\mathscr{F}$ is not $>_q$-merged-log-concave, since the following is not a q-polynomial:

$$\Delta(\mathscr{F})(s,l,w,(3,3,3),(3,4,3),(0,0,0,1,0,1),x,\mathfrak{X}) = q^{\frac{37}{2}} + 3q^{\frac{35}{2}} + 7q^{\frac{33}{2}} + 13q^{\frac{31}{2}} + 19q^{\frac{29}{2}} + 23q^{\frac{27}{2}} + 24q^{\frac{25}{2}} + 23q^{\frac{23}{2}} + 20q^{\frac{21}{2}} + 17q^{\frac{19}{2}} + 12q^{\frac{17}{2}} + 7q^{\frac{15}{2}} + 3q^{\frac{13}{2}} + q^{\frac{11}{2}}.$$

However, ζ is ideal, because each $i \in [\![\theta]\!]$ gives

$$\begin{aligned}
&\Delta(\mathscr{F})(s,l,w,m_i,n_i,k_i,x,\mathfrak{X}) \\
&= \frac{(4i+2)_q(4i+2)_q(4i+2)_q(4i+6)_q(4i+6)_q(4i+6)_q}{(0)_q(1)_q(2)_q(4)_q(5)_q(6)_q} \\
&\cdot \left(\left(\frac{q^{\frac{(4i+2)^2-(4i+1)^2+(4i)^2}{2}}}{(4i+2)_q(4i+1)_q(4i)_q} \right)^2 \right. \\
&\left. - \frac{q^{\frac{(4i-2)^2-(4i-3)^2+(4i-4)^2}{2}}}{(4i-2)_q(4i-3)_q(4i-4)_q} \cdot \frac{q^{\frac{(4i+6)^2-(4i+5)^2+(4i+4)^2}{2}}}{(4i+6)_q(4i+5)_q(4i+4)_q} \right)
\end{aligned}$$

such that

$$\begin{aligned}
\left(q^{\frac{(4i+2)^2-(4i+1)^2+(4i)^2}{2}} \right)^2 &= q^{16i^2+8i+3} >_q 0, \\
q^{\frac{(4i-2)^2-(4i-3)^2+(4i-4)^2}{2}} \cdot q^{\frac{(4i+6)^2-(4i+5)^2+(4i+4)^2}{2}} &= q^{16i^2+8i+19} >_q 0.
\end{aligned}$$

14.6.3 Polynomials with positive integer coefficients of an ideal merged pair

For instance, $\Delta(\mathscr{F})(s,l,w,m_0,n_0,k_0,x,\mathfrak{X})$ is

$$q^3+3q^4+6q^5+10q^6+15q^7+20q^8+23q^9+24q^{10}+23q^{11}+20q^{12}+15q^{13}+10q^{14}+6q^{15}+3q^{16}+q^{17}.$$

Then, $\Delta(\mathscr{F})(s,l,w,m_1,n_1,k_1,x,\mathfrak{X})$ is

$$\begin{aligned}
&q^{27} + 5q^{28} + 19q^{29} + 58q^{30} + 158q^{31} + 388q^{32} + 885q^{33} + 1890q^{34} \\
&+ 3828q^{35} + 7390q^{36} + 13688q^{37} + 24412q^{38} + 42089q^{39} + 70327q^{40} \\
&+ 114182q^{41} + 180469q^{42} + 278185q^{43} + 418794q^{44} + 616576q^{45} \\
&+ 888721q^{46} + 1255398q^{47} + 1739429q^{48} + 2365848q^{49} + 3160960q^{50} \\
&+ 4151230q^{51} + 5361659q^{52} + 6814044q^{53} + 8524865q^{54} + 10503235q^{55} \\
&+ 12748773q^{56} + 15249837q^{57} + 17982091q^{58} + 20907732q^{59} + 23975445q^{60} \\
&+ 27121214q^{61} + 30270096q^{62} + 33338786q^{63} + 36239100q^{64} + 38881893q^{65} \\
&+ 41181562q^{66} + 43060365q^{67} + 44452792q^{68} + 45309075q^{69} + 45598218q^{70} \\
&+ 45309614q^{71} + 44453850q^{72} + 43061901q^{73} + 41183518q^{74} + 38884194q^{75} \\
&+ 36241661q^{76} + 33341512q^{77} + 30272892q^{78} + 27123986q^{79} + 23978109q^{80} \\
&+ 20910214q^{81} + 17984335q^{82} + 15251804q^{83} + 12750444q^{84} + 10504608q^{85} \\
&+ 8525955q^{86} + 6814878q^{87} + 5362272q^{88} + 4151661q^{89} + 3161248q^{90} \\
&+ 2366030q^{91} + 1739536q^{92} + 1255456q^{93} + 888749q^{94} + 616588q^{95} \\
&+ 418798q^{96} + 278186q^{97} + 180469q^{98} + 114182q^{99} + 70327q^{100} \\
&+ 42089q^{101} + 24412q^{102} + 13688q^{103} + 7390q^{104} + 3828q^{105} + 1890q^{106} \\
&+ 885q^{107} + 388q^{108} + 158q^{109} + 58q^{110} + 19q^{111} + 5q^{112} + q^{113}.
\end{aligned}$$

Also, $\Delta(\mathscr{F})(s,l,w,m_2,n_2,k_2,x,\mathfrak{X})$ is

$$q^{83} + 5q^{84} + 19q^{85} + 58q^{86} + 158q^{87} + 390q^{88} + 899q^{89} + 1951q^{90} + \dots.$$

14.6.4 Conjectures

Some leading coefficients of $\Delta(\mathscr{F})(s,l,w,m_i,n_i,k_i,x,\mathfrak{X})$ for $i \in [\![2]\!]$ coincide in the above. Also, these $\Delta(\mathscr{F})(s,l,w,m_i,n_i,k_i,x,\mathfrak{X})$ are not palindromic, but log-concave q-polynomials. Thus, we make the following conjectures.

Conjecture 14.3. *For each $i \in \mathbb{Z}_{\geq 0}$, consider the q-polynomial*

$$f_i = \sum_j f_{i,j} q^j = \Delta(\mathscr{F})(s,l,w,m_i,n_i,k_i,x,\mathfrak{X}).$$

For $\operatorname{ord}_q(f_i) < j_2 < \cdots < \deg_q(f_i)$, let $f_i^+ = \left(f_{i,\operatorname{ord}_q(f_i)}, f_{i,j_2}, \ldots, f_{i,\deg_q(f_i)}\right)$ denote the non-zero positive coefficients of f_i. Then, each $i \in \mathbb{Z}_{\geq 1}$ satisfies

$$f_{i+1}^+(2+4(i-1), 5+4(i-1)) = f_i^+(2+4(i-1), 5+4(i-1)).$$

For example, $f_1^+ = (1,5,19,58,158,388,885,\ldots)$ and $f_2^+ = (1,5,19,58,158,390,899,\ldots)$ in Section 14.6.3. Thus, we have

$$f_1^+(2,5) = (5,19,58,158) = f_2^+(2,5).$$

We adopt the following notation to state another conjecture on log-concave q-polynomials.

Definition 14.4. *For a Laurent polynomial $f \in \mathbb{Q}[q^{\pm 1}]$, let*

$$C_q(f) = \begin{cases} q^{-\operatorname{ord}_q(f)} f \text{ if } f \neq 0, \\ 0 \text{ otherwise}. \end{cases}$$

Conjecture 14.5. *Let $i \in \mathbb{Z}_{\geq 0}$.*

1. *$\Delta(\mathscr{F})(s,l,w,m_i,n_i,k_i,x,\mathfrak{X})$ is a log-concave q-polynomial.*
2. *The following is a log-concave q-polynomial:*

$$C_q(\Delta(\mathscr{F})(s,l,w,m_{i+1},n_{i+1},k_{i+1},x,\mathfrak{X})) - C_q(\Delta(\mathscr{F})(s,l,w,m_i,n_i,k_i,x,\mathfrak{X})) >_q 0.$$

Claim 1 of Conjecture 14.5 is analogous to Conjectures 8.54 and 8.56.

15 Parcel convolutions

15.1 Convolution indices

Consider a parcel $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ with $l=1$ and $s=(0,\infty)$. Then, multiplying generating functions of $\mathscr{F}$ gives a sequence of rational functions, which, however, is not necessarily a parcel $\Lambda(s,l,w,\succ,g_s,\phi,\rho,x,\mathfrak{X})$ for some $g_s = (g_{s,m} \in \mathbb{Q}(\mathfrak{X}))_{m \in \mathbb{Z}^l}$. Hence, we introduce the notion of convolution indices.

Definition 15.1. *Assume the following:*

(a) squaring orders $O_i = \{\succeq_i, \succ_i\}$ on $\mathfrak{X}$ for $i \in [\![3]\!]$ such that $O_3 \supseteq O_1, O_2$;

(b) $l=1$, $\lambda \in \mathbb{Q}_{>0}^3$, and $x = \left((q^{\lambda_i})\right)_{i \in [\![3]\!]} \in \prod_{i \in [\![3]\!]} \mathbb{Q}(\mathfrak{X})^l$;

(c) $\mathscr{F}_i = \Lambda(s_i, l, w, \succ_i, f_{i,s_i}, \phi, \rho_i, x_i, \mathfrak{X})$ *for* $i \in [\![2]\!]$;

(d) $\rho = (\rho_i)_{i \in [\![3]\!]} \in \prod_{i \in [\![3]\!]} \mathbb{Z}^l$, $s = (s_1, s_2, s_1 + s_2)$, $O = (O_i)_{i \in [\![3]\!]}$, *and* $o \in \mathbb{Z}^2_{\geq 1}$.

We call $C = (s, l, w, O, \phi, \rho, x, \mathfrak{X}, q, \lambda, o)$ *a convolution index of* $(\mathscr{F}_1, \mathscr{F}_2)$ *when* C *satisfies the following conditions:*

1. $o_1 \lambda_1 = o_2 \lambda_2 = \lambda_3$;
2. $o_1^{-1} \rho_1 = o_2^{-1} \rho_2 = \rho_3$;
3. ϕ *is a* $(s_i, l, w, \succ_i, (o_i), x_i, \mathfrak{X})$*-mediator for each* $i \in [\![2]\!]$;
4. ϕ *is a* $(s_3, l, w, \succ_3, \rho_3, x_3, \mathfrak{X})$*-mediator.*

We refer to Conditions 1, 2, 3, and 4 as the (λ, o)*-exponent equation,* (ρ, o)*-base-shift equation,* (C, o)*-mediator condition, and* (C, ρ_3)*-mediator condition.*

In particular, we have the following on convolution indices.

Lemma 15.2. *Suppose a convolution index* $C = (s, l, w, O, \phi, \rho, x, \mathfrak{X}, q, \lambda, o)$ *of* $(\mathscr{F}_1, \mathscr{F}_2)$.

1. *Then,* x_i *is* O_i*-admissible for each* $i \in [\![2]\!]$.
2. *Provided the* (λ, o)*-exponent equation,* x_3 *is* O_3*-admissible.*
3. *Suppose that* λ *and* ρ *are flat.*
   (a) *If* $o = \iota^2(1)$*, then we have the* (λ, o)*-exponent and* (ρ, o)*-base-shift equations and the* (C, o)*-mediator condition.*
   (b) *If one of* s_1 *and* s_2 *is infinite or* $(0,0)$*, then we have the* (C, ρ_3)*-mediator condition.*
   (c) *Similarly, if* $w = (0)$*, then we have the* (C, ρ_3)*-mediator condition.*
4. *Suppose that* ϕ *is the canonical* l*-mediator.*
   (a) *Then, the* (C, o)*-mediator condition holds.*
   (b) *Provided the* (λ, o)*-exponent equation, the* (C, ρ_3)*-mediator condition holds.*

Proof. Proof of Claim 1. Claim 1 holds, since the parcel $\mathscr{F}_i$ has the squaring order O_i and the base x_i.

Proof of Claim 2. We have $o \in \mathbb{Z}^2_{\geq 1}$ for the convolution index C. Then, since there is the (λ, o)-exponent equation, $x_3 = \left(q^{\lambda_3}\right)$ is O_1- and O_2-admissible by Claim 2 of Proposition 5.24 and Claim 1. Thus, Claim 2 holds by the compatibility $O_3 \Supset O_1, O_2$ in C.

Proof of Claim 3a. By $o = \iota^2(1)$, the flatness of λ and ρ yields the (λ, o)-exponent and (ρ, o)-base-shift equations. Let $\mu_i = (s_i, l, w, \succ_i, (o_i), x_i, \mathfrak{X})$ for $i \in [\![2]\!]$. By Claim 1, Claim 2 of Lemma 5.20 implies $1 \succ_i 0$ for $i \in [\![2]\!]$. This gives the base-shift positivity of ϕ and μ_i by Claim 1 of Lemma 4.2 and $o = \iota^2(1)$. Thus, the (C, o)-mediator condition holds, since we have the base positivity of ϕ and μ_i in the parcel $\mathscr{F}_i$.

Proof of Claim 3b. Suppose that s_1 is infinite or $(0,0)$. Then, $m \in [\![s_1 + s_2]\!]^l$ implies $m \in [\![s_1]\!]^l$ if s_1 is infinite, or $m \in [\![s_2]\!]^l$ otherwise. Thus, by the flatness of ρ and λ, $m \in [\![s_1 + s_2]\!]^l$ such that $m \in [\![s_i]\!]^l$ gives

$$B(s_1 + s_2, l, w, m, \phi, \rho_3, x_3, \mathfrak{X}) = B(s_i, l, w, m, \phi, \rho_i, x_i, \mathfrak{X}) \succ_i 0.$$

Therefore, the (C,ρ_3)-mediator condition holds by the compatibility $O_3 \Supset O_1, O_2$ in C.

Proof of Claim 3c. Let $\mu_3 = (s_3, l, w, \succ_3, \rho_3, x_3, \mathfrak{X})$. Notice that x_3 is $\succ_3$-admissible by the flat λ and the compatibility $O_3 \Supset O_1$. First, Claim 1 and Claim 2 of Lemma 5.20 imply $1 \succ_1 0$, which gives $1 \succ_3 0$ by the compatibility. Hence, the base-shift positivity of ϕ and μ_3 follows from Claim 1 of Lemma 4.2. Moreover, Claim 3c follows, since we have the base positivity of ϕ and μ_3 by $w = (0)$.

Proof of Claim 4a. Claim 4a follows from Claim 1, Claim 2 of Lemma 5.20, and Lemma 1.8.

Proof of Claim 4b. Similarly, Claim 4b follows from Claim 2, Claim 2 of Lemma 5.20, and Lemma 1.8.

□

We now obtain parcels by convolution indices.

Proposition 15.3. *Consider $\mathscr{F}_i = \Lambda(s_i, l, w, \succ_i, f_{i,s_i}, \phi, \rho_i, x_i, \mathfrak{X})$ for $i \in [\![2]\!]$ with a convolution index $C = (s, l, w, O, \phi, \rho, x, \mathfrak{X}, q, \lambda, o)$. Assume*

$$\mathscr{H} = \left(\mathscr{H}_{m_1} = \sum_{m_2 \in \mathbb{Z}^l} \mathscr{F}_{1,m_2} \cdot \mathscr{F}_{2,m_1-m_2} \in \mathbb{Q}(\mathfrak{X})\right)_{m_1 \in \mathbb{Z}^l},$$

$$f_{3,s_3} = \left(f_{3,s_3,m} = \prod \phi(x_3)^{m \circ w} \cdot [m]!^w_{x_3} \cdot \mathscr{H}_m \in \mathbb{Q}(\mathfrak{X})\right)_{m \in \mathbb{Z}^l}.$$

Let $\tau_i = (o_i) \in \mathbb{Z}^l_{\geq 1}$ for $i \in [\![2]\!]$. Then, we have the following.

1. *When $m \notin [\![s_3]\!]^l$,*

$$\mathscr{H}_m = 0.$$

2. *When $m_1 \in \mathbb{Z}^l$,*

$$f_{3,s_3,m_1} = \sum_{m_2 \in \mathbb{Z}^l} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}^w_{x_3} \cdot B(s_1, l, w, m_2, \phi, \tau_1, x_1, \mathfrak{X}) \cdot B(s_2, l, w, m_1 - m_2, \phi, \tau_2, x_2, \mathfrak{X}) \cdot f_{1,s_1,m_2} f_{2,s_2,m_1-m_2}.$$

3. *f_{3,s_3} is $\succ_3$-positive.*

4. *$\mathscr{H}$ is a parcel such that*

$$\mathscr{H} = \Lambda(s_3, l, w, \succ_3, f_{3,s_3}, \phi, \rho_3, x_3, \mathfrak{X}).$$

Proof. Proof of Claim 1. Claim 1 holds, because $\mathscr{F}_{1,m} = 0$ for $m \notin [\![s_1]\!]^l$ and $\mathscr{F}_{2,m} = 0$ for $m \notin [\![s_2]\!]^l$.

Proof of Claim 2. Since $x_3 = \left(q^{\lambda_3}\right)$, we have

$$x_3 = \left(q^{o_1 \lambda_1}\right) = x_1^{\tau_1},$$
$$x_3 = \left(q^{o_2 \lambda_2}\right) = x_2^{\tau_2}$$

by the (λ, o)-exponent equation. Thus, for each $i \in [\![2]\!]$ and $m \in [\![s_i]\!]^l$, we have

$$\frac{\prod \phi(x_3)^{m \circ w} \cdot [m]!^w_{x_3}}{\prod \phi(x_i)^{m \circ w} \cdot [m]!^w_{x_i}} = \frac{\prod \phi(x_i^{\tau_i})^{m \circ w} \cdot [m]!^w_{x_i^{\tau_i}}}{\prod \phi(x_i)^{m \circ w} \cdot [m]!^w_{x_i}} = B(s_i, l, w, m, \phi, \tau_i, x_i, \mathfrak{X}).$$

Hence, if $m_1 \in [\![s_3]\!]^l$, then

$$
\begin{aligned}
f_{3,s_3,m_1} &= \prod \phi(x_3)^{m_1 \circ w} \cdot [m_1]!^w_{x_3} \cdot \sum_{m_2 \in \mathbb{Z}^l} \mathscr{F}_{1,m_2} \mathscr{F}_{2,m_1-m_2} \\
&= \prod \phi(x_3)^{m_1 \circ w} \cdot [m_1]!^w_{x_3} \\
&\cdot \sum_{m_2 \in [\![s_1]\!]^l, m_1-m_2 \in [\![s_2]\!]^l} \frac{f_{1,s_1,m_2}}{\prod \phi(x_1)^{m_2 \circ w} \cdot [m_2]!^w_{x_1}} \cdot \frac{f_{2,s_2,m_1-m_2}}{\prod \phi(x_2)^{(m_1-m_2) \circ w} \cdot [m_1-m_2]!^w_{x_2}} \\
&= \sum_{m_2 \in [\![s_1]\!]^l, m_1-m_2 \in [\![s_2]\!]^l} \frac{[m_1]!^w_{x_3}}{[m_2]!^w_{x_3} [m_1-m_2]!^w_{x_3}} \\
&\cdot \frac{\prod \phi(x_3)^{m_2 \circ w} \cdot [m_2]!^w_{x_3}}{\prod \phi(x_1)^{m_2 \circ w} \cdot [m_2]!^w_{x_1}} \cdot \frac{\prod \phi(x_3)^{(m_1-m_2) \circ w} \cdot [m_1-m_2]!^w_{x_3}}{\prod \phi(x_2)^{(m_1-m_2) \circ w} \cdot [m_1-m_2]!^w_{x_2}} \cdot f_{1,s_1,m_2} f_{2,s_2,m_1-m_2} \\
&= \sum_{m_2 \in [\![s_1]\!]^l, m_1-m_2 \in [\![s_2]\!]^l} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}^w_{x_3} \\
&\cdot B(s_1,l,w,m_2,\phi,\tau_1,x_1,\mathfrak{X}) \cdot B(s_2,l,w,m_1-m_2,\phi,\tau_2,x_2,\mathfrak{X}) \cdot f_{1,s_1,m_2} f_{2,s_2,m_1-m_2}.
\end{aligned}
$$

If $m_1 \notin [\![s_3]\!]^l$, then both sides of Claim 2 are zero by Claim 1 and the $\succ_i$-positivities of f_{i,s_i} of $1 \leq i \leq 2$.

Proof of Claim 3. For $m_1 \in [\![s_3]\!]^l$, we obtain

$$
\begin{aligned}
f_{3,s_3,m_1} = &\sum_{m_2 \in [\![s_1]\!]^l, m_1-m_2 \in [\![s_2]\!]^l} \\
&\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}^w_{x_3} \cdot B(s_1,l,w,m_2,\phi,\tau_1,x_1,\mathfrak{X}) \cdot B(s_2,l,w,m_1-m_2,\phi,\tau_2,x_2,\mathfrak{X}) \\
&\cdot f_{1,s_1,m_2} \cdot f_{2,s_2,m_1-m_2}.
\end{aligned} \tag{15.1.1}
$$

In equation (15.1.1), $m_1, m_1 - m_2 \geq 0$. This gives $\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}^w_{x_3} \succ_3 0$ by the $\succ_3$-admissibility of x_3 in Claim 2 of Lemma 15.2. Also, in equation (15.1.1), the compatibility $\succ_3 \supseteq \succ_1, \succ_2$ implies

$$
\begin{aligned}
B(s_1,l,w,m_2,\phi,\tau_1,x_1,\mathfrak{X}) \cdot B(s_2,l,w,m_1-m_2,\phi,\tau_2,x_2,\mathfrak{X}) &\succ_3 0, \\
f_{1,s_1,m_2} \cdot f_{2,s_2,m_1-m_2} &\succ_3 0
\end{aligned}
$$

by the $\succ_i$-positivities of f_{i,s_i} for $1 \leq i \leq 2$. Hence, Claim 3 holds by equation (15.1.1), since $f_{3,s_3,n} = 0$ for $n \notin [\![s_3]\!]^l$ by Claim 1.

Proof of Claim 4. Claim 4 follows from Claim 3 and the (C, ρ_3)-mediator condition. □

By Proposition 15.3, we introduce the parcel convolutions below.

Definition 15.4. *Suppose $\mathscr{F}_i = \Lambda(s_i, l, w, \succ_i, f_{i,s_i}, \phi, \rho_i, x_i, \mathfrak{X})$ for $i \in [\![2]\!]$ with a convolution index $C = (s, l, w, O, \phi, \rho, x, \mathfrak{X}, q, \lambda, o)$. Then, we define the parcel convolution*

$$\mathscr{F}_1 * \mathscr{F}_2 = \Lambda(s_3, l, w, \succ_3, f_{3,s_3}, \phi, \rho_3, x_3, \mathfrak{X})$$

such that $f_{3,s_3,m} = \prod \phi(x_3)^{m \circ w} \cdot [m]!^w_{x_3} \cdot \sum_{n_1+n_2=m} \mathscr{F}_{1,n_1} \mathscr{F}_{2,n_2} \in \mathbb{Q}(\mathfrak{X})$ for $m \in \mathbb{Z}^l$.

In particular, we have the following parcels by identity functions.

Lemma 15.5. *Suppose* $s_1 = (0,0)$, $l = 1$, $\mathscr{F} = \Lambda(s_1, l, w, \succ, 1_{s_1}, \phi, \rho, x, \mathfrak{X})$, *and* $\mathscr{G} = \Lambda(s_2, l, w, \succ, f_{s_2}, \phi, \rho, x, \mathfrak{X})$. *Then,* $\mathscr{F} * \mathscr{G} = \mathscr{G}$.

Proof. Let $x = (q)$ and $O = (O_i = \{\succeq, \succ\})_{i \in [\![3]\!]}$. Then, $(\mathscr{F}, \mathscr{G})$ has the convolution index

$$\left((s_1, s_2, s_2), l, w, O, \phi, \iota^3(\rho), \iota^3(x), \mathfrak{X}, q, \iota^3(1), \iota^2(1)\right)$$

by Claims 3a and 3b of Lemma 15.2. Thus, the assertion holds by $\mathscr{F}_{(0)} = \frac{1_{s',(0)}}{\prod \phi(x)^{(0) \circ w} \cdot [(0)]!^w_x} = 1$. □

Hence, we have the convolutions below by Lemma 15.5 and Claim 3 of Lemma 15.2.

Definition 15.6. *Let* $l = 1$, $\lambda \in \mathbb{Z}_{\geq 0}$, *and* $\mathscr{F} = \Lambda(s, l, w, \succ, f_s, \phi, \rho, x, \mathfrak{X})$. *Assume that* $s = (0,0)$ *or infinite, or* $w = (0)$. *Then, we define the* λ*-fold parcel convolution*

$$\mathscr{F}^{*\lambda} = \Lambda(\lambda s, l, w, \succ, g_{\lambda s}, \phi, \rho, x, \mathfrak{X})$$

such that $g_{\lambda s} = 1_{(0,0)}$ *when* $\lambda = 0$, *and*

$$g_{\lambda s, m} = \prod \phi(x)^{m \circ w} \cdot [m]!^w_x \cdot \sum_{n_1 + \cdots + n_\lambda = m} \mathscr{F}_{n_1} \cdot \cdots \cdot \mathscr{F}_{n_\lambda} \text{ for } m \in \mathbb{Z}^l \text{ when } \lambda > 0.$$

15.2 Extension of the Cauchy–Binet formula

The Cauchy–Binet formula describes minors of a matrix product AB by those of A and B. Hence, we extend the Cauchy–Binet formula to obtain the merged-log-concavity of parcel convolutions, as merged determinants extend 2×2 determinants. This uses the following notation with Definition 2.3.

Definition 15.7. *Let* $d \in \mathbb{Z}_{\geq 1}$, $\lambda \in \mathbb{Z}^3_{\geq 1}$, *and* Q *be a commutative ring.*

1. *Consider a* $\lambda_1 \times \lambda_2$*-matrix* $A = (A_{i,j})_{i \in [\![\lambda_1]\!], j \in [\![\lambda_2]\!]} \in M_{\lambda_1, \lambda_2}(Q)$. *Suppose* $\alpha \in \mathrm{T}_<(d, \lambda_1)$ *and* $\beta \in \mathrm{T}_<(d, \lambda_2)$. *Then, we write the minor*

$$A(\alpha, \beta) = \det \begin{bmatrix} A_{\alpha_1,\beta_1} & A_{\alpha_1,\beta_2} & \cdots & A_{\alpha_1,\beta_d} \\ A_{\alpha_2,\beta_1} & A_{\alpha_2,\beta_2} & \cdots & A_{\alpha_2,\beta_d} \\ \cdots & \cdots & \cdots & \cdots \\ A_{\alpha_d,\beta_1} & A_{\alpha_d,\beta_2} & \cdots & A_{\alpha_d,\beta_d} \end{bmatrix}.$$

2. *Suppose* $\alpha \in \mathrm{T}_<(d, \lambda_1)$, $\beta \in \mathrm{T}_<(d, \lambda_3)$. $A \in M_{\lambda_1, \lambda_2}(Q)$, *and* $B \in M_{\lambda_2, \lambda_3}(Q)$. *Then, let*

$$\mathrm{T}_<(d, \lambda_2, \alpha, \beta, A, B) = \{\gamma \in \mathrm{T}_<(d, \lambda_2) \mid A(\alpha, \gamma) B(\gamma, \beta) \neq 0\}.$$

We recall the Cauchy–Binet formula, which holds by summation reordering in the Leibniz formula of $A(\alpha, \gamma)$ and $B(\gamma, \beta)$.

Theorem 15.8 (the Cauchy–Binet formula)**.** *Let* $d \in \mathbb{Z}_{\geq 1}$, $\lambda \in \mathbb{Z}^3_{\geq 1}$, *and* Q *be a commutative ring. Consider* $A \in M_{\lambda_1, \lambda_2}(Q)$, $B \in M_{\lambda_2, \lambda_3}(Q)$, $\alpha \in \mathrm{T}_<(d, \lambda_1)$, *and* $\beta \in \mathrm{T}_<(d, \lambda_3)$. *Then,* $AB(\alpha, \beta) = \sum_{\gamma \in \mathrm{T}_<(d, \lambda_2)} A(\alpha, \gamma) B(\gamma, \beta)$.

Let $Q^\times$ denote the units of a ring Q. We then generalize the ring shift factors in Definition 6.1.

Definition 15.9. *Let* Q *be a commutative ring and* $d \in \mathbb{Z}_{\geq 1}$. *Let* $\Theta(\kappa, u) \in Q$ *for* $\kappa \in [\![3]\!]$ *and* $u \in \mathbb{Z}$.

1. *For $i \in [\![2]\!]$, suppose $\phi_i \in \mathbb{Z}_{\geq 1}$, $\gamma_i \in \mathrm{T}_<(d, \phi_i)$, and $\mu_i \in [\![3]\!]$.*

   (a) *For each $u \in \mathbb{Z}$, we define the ring shift factor*

$$\Theta(\mu_1, \mu_2, \gamma_1, \gamma_2, u) = \begin{cases} \dfrac{\prod_{i \in [\![d]\!]} \Theta(\mu_2, \gamma_{2,i} - \gamma_{1,1} + u)}{\prod_{i \in [\![d]\!]} \Theta(\mu_1, \gamma_{1,i} - \gamma_{1,1} + u)} \text{ if } \prod_{i \in [\![d]\!]} \Theta(\mu_1, \gamma_{1,i} - \gamma_{1,1} + u) \in Q^\times, \\ 0 \text{ otherwise}. \end{cases}$$

   (b) *For each $u \in \mathbb{Z}$ and $C \in M_{\phi_1,\phi_2}(Q)$, let*

$$C(\mu_1, \mu_2, \gamma_1, \gamma_2, \Theta, u) = \Theta(\mu_1, \mu_2, \gamma_1, \gamma_2, u) \cdot C(\gamma_1, \gamma_2).$$

   If $\Theta(1,u) = \Theta(2,u) = \Theta(3,u)$ for each $u \in \mathbb{Z}$, then we also write $\Theta(u) = \Theta(\kappa, u)$ for $\kappa \in [\![3]\!]$, $\Theta(\gamma_1, \gamma_2, u) = \Theta(\mu_1, \mu_2, \gamma_1, \gamma_2, u)$, and $C(\gamma_1, \gamma_2, \Theta, u) = C(\mu_1, \mu_2, \gamma_1, \gamma_2, \Theta, u)$.

2. *Suppose $\lambda \in \mathbb{Z}^3_{\geq 1}$, $A \in M_{\lambda_1,\lambda_2}(Q)$, $B \in M_{\lambda_2,\lambda_3}(Q)$, $\alpha \in \mathrm{T}_<(d, \lambda_1)$, and $\beta \in \mathrm{T}_<(d, \lambda_3)$. Then, we call $R = (\Theta, d, \lambda, \alpha, \beta, A, B, Q)$ a ring shift index when R satisfies the following conditions:*

   (a) *$\Theta(\kappa, u) \in Q^\times$ for each $\kappa \in [\![3]\!]$ and $u \geq 0$;*

   (b) *$\prod_{i \in [\![d]\!]} \Theta(\kappa, \gamma_i - \alpha_1 + u) \in Q^\times$ for each $\kappa \in [\![3]\!]$, $\gamma \in \mathrm{T}_<(d, \lambda_2, \alpha, \beta, A, B)$, and $u \geq 0$;*

   (c) *$\prod_{i \in [\![d]\!]} \Theta(\kappa, \beta_i - \gamma_1 + u) \in Q^\times$ for each $\kappa \in [\![2,3]\!]$, $\gamma \in \mathrm{T}_<(d, \lambda_2, \alpha, \beta, A, B)$, and $u \geq \min(0, \gamma_1 - \alpha_1)$.*

   We call Conditions 2a, 2b, and 2c the Θ-nonsingularity, $[\![3]\!]$-product nonsingularity, and $[\![2,3]\!]$-product nonsingularity of R, respectively.

There is a ring shift index $(\Theta, d, \lambda, \alpha, \beta, A, B, Q)$ such that each $\Theta(u) = 1$. Then, $AB(\alpha, \beta, \Theta, u) = AB(\alpha, \beta)$, $A(\alpha, \gamma, \Theta, u) = A(\alpha, \gamma)$, and $B(\gamma, \beta, \Theta, u) = B(\gamma, \beta)$. Hence, ring shift indices extend minors by ring shift factors.

We state the following lemma.

Lemma 15.10. *Consider a ring shift index $(\Theta, d, \lambda, \alpha, \beta, A, B, Q)$. Let $\gamma \in \mathrm{T}_<(d, \lambda_2, \alpha, \beta, A, B)$. Then, we have the following.*

1. *$\Theta(\mu_1, \mu_2, \alpha, \gamma, u) \in Q^\times$ whenever $\mu_1, \mu_2 \in [\![3]\!]$ and $u \geq 0$.*

2. *$\Theta(\mu_1, \mu_2, \gamma, \beta, u) \in Q^\times$ whenever $\mu_1, \mu_2 \in [\![2,3]\!]$ and $u \geq \min(0, \gamma_1 - \alpha_1)$.*

Proof. Proof of Claim 1. Claim 1 holds, since we have

$$\prod_{i \in [\![d]\!]} \Theta(\mu_1, \alpha_i - \alpha_1 + u) \in Q^\times,$$
$$\prod_{i \in [\![d]\!]} \Theta(\mu_2, \gamma_i - \alpha_1 + u) \in Q^\times$$

by the Θ- and $[\![3]\!]$-product nonsingularities.

Proof of Claim 2. If $0 = \min(0, \gamma_1 - \alpha_1)$, then

$$\prod_{i \in [\![d]\!]} \Theta(\mu_1, \gamma_i - \gamma_1 + u) \in Q^\times$$

for each $u \geq \min(0, \gamma_1 - \alpha_1)$ by the Θ–nonsingularity; otherwise, the same holds by the $[\![3]\!]$–product nonsingularity. Thus, Claim 2 follows from the $[\![2,3]\!]$-product nonsingularity. □

Now, we obtain the following extended Cauchy–Binet formula by the ring shift factors.

Theorem 15.11 (the extended Cauchy–Binet formula by ring shift factors)**.** *Consider a ring shift index* $(\Theta, d, \lambda, \alpha, \beta, A, B, Q)$*. Then, for each* $u \in \mathbb{Z}_{\geq 0}$*, we have*

$$AB(1,3,\alpha,\beta,\Theta,u) = \sum_{\gamma \in \mathrm{T}_{<}(d,\lambda_2)} A(1,2,\alpha,\gamma,\Theta,u) B(2,3,\gamma,\beta,\Theta,\gamma_1 - \alpha_1 + u).$$

Proof. Let $\gamma \in \mathrm{T}_{<}(d,\lambda_2)$ and $u \geq 0$. First, suppose $A(\alpha,\gamma)B(\gamma,\beta) = 0$. Then,

$$\begin{aligned} & A(1,2,\alpha,\gamma,\Theta,u)B(2,3,\gamma,\beta,\Theta,\gamma_1 - \alpha_1 + u) \\ &= \Theta(1,2,\alpha,\gamma,u)A(\alpha,\gamma)\Theta(2,3,\gamma,\beta,\gamma_1 - \alpha_1 + u)B(\gamma,\beta) \\ &= 0. \end{aligned} \tag{15.2.1}$$

Second, suppose $A(\alpha,\gamma)B(\gamma,\beta) \neq 0$. Then, Lemma 15.10 yields

$$\begin{aligned} A(\alpha,\gamma) \cdot B(\gamma,\beta) = {} & \Theta(1,2,\alpha,\gamma,u)^{-1} \cdot \Theta(2,3,\gamma,\beta,\gamma_1 - \alpha_1 + u)^{-1} \\ & \cdot A(1,2,\alpha,\gamma,\Theta,u) \cdot B(2,3,\gamma,\beta,\Theta,\gamma_1 - \alpha_1 + u). \end{aligned} \tag{15.2.2}$$

Moreover, we have

$$\begin{aligned} & \Theta(1,3,\alpha,\beta,u) \cdot \Theta(1,2,\alpha,\gamma,u)^{-1} \cdot \Theta(2,3,\gamma,\beta,\gamma_1 - \alpha_1 + u)^{-1} \\ &= \frac{\prod_{i \in [\![d]\!]} \Theta(3, \beta_i - \alpha_1 + u)}{\prod_{i \in [\![d]\!]} \Theta(1, \alpha_i - \alpha_1 + u)} \cdot \frac{\prod_{i \in [\![d]\!]} \Theta(1, \alpha_i - \alpha_1 + u)}{\prod_{i \in [\![d]\!]} \Theta(2, \gamma_i - \alpha_1 + u)} \cdot \frac{\prod_{i \in [\![d]\!]} \Theta(2, \gamma_i - \gamma_1 + \gamma_1 - \alpha_1 + u)}{\prod_{i \in [\![d]\!]} \Theta(3, \beta_i - \gamma_1 + \gamma_1 - \alpha_1 + u)} \\ &= \frac{\prod_{i \in [\![d]\!]} \Theta(3, \beta_i - \alpha_1 + u)}{\prod_{i \in [\![d]\!]} \Theta(1, \alpha_i - \alpha_1 + u)} \cdot \frac{\prod_{i \in [\![d]\!]} \Theta(1, \alpha_i - \alpha_1 + u)}{\prod_{i \in [\![d]\!]} \Theta(2, \gamma_i - \alpha_1 + u)} \cdot \frac{\prod_{i \in [\![d]\!]} \Theta(2, \gamma_i - \alpha_1 + u)}{\prod_{i \in [\![d]\!]} \Theta(3, \beta_i - \alpha_1 + u)} \\ &= 1. \end{aligned} \tag{15.2.3}$$

Therefore, equations (15.2.1), (15.2.2), and (15.2.3) and Theorem 15.8 yield

$$\begin{aligned} & AB(1,3,\alpha,\beta,\Theta,u) \\ &= \Theta(1,3,\alpha,\beta,u) \cdot AB(\alpha,\beta) \\ &= \sum_{\gamma \in \mathrm{T}_{<}(d,\lambda_2)} \Theta(1,3,\alpha,\beta,u) \cdot A(\alpha,\gamma)B(\gamma,\beta) \\ &= \sum_{\gamma \in \mathrm{T}_{<}(d,\lambda_2,\alpha,\beta,A,B)} \Theta(1,3,\alpha,\beta,u) \cdot \Theta(1,2,\alpha,\gamma,u)^{-1} \cdot \Theta(2,3,\gamma,\beta,\gamma_1 - \alpha_1 + u)^{-1} \\ & \cdot A(1,2,\alpha,\gamma,\Theta,u) \cdot B(2,3,\gamma,\beta,\Theta,\gamma_1 - \alpha_1 + u) \\ &= \sum_{\gamma \in \mathrm{T}_{<}(d,\lambda_2,\alpha,\beta,A,B)} A(1,2,\alpha,\gamma,\Theta,u)B(2,3,\gamma,\beta,\Theta,\gamma_1 - \alpha_1 + u) \\ &= \sum_{\gamma \in \mathrm{T}_{<}(d,\lambda_2)} A(1,2,\alpha,\gamma,\Theta,u)B(2,3,\gamma,\beta,\Theta,\gamma_1 - \alpha_1 + u). \end{aligned}$$

□

In particular, the extended Cauchy–Binet formula reduces to the non-extended one with trivial ring shift factors

$$1 = \Theta(1,3,\alpha,\beta,u) = \Theta(1,2,\alpha,\gamma,u) = \Theta(2,3,\gamma,\beta,\gamma_1 - \alpha_1 + u)$$

of each $\gamma \in \mathrm{T}_{<}(d,\lambda_2,\alpha,\beta,A,B)$ and $u \geq 0$.

Remark 15.12. If $\Theta(u)=[u]!_q$ for each $u\geq 0$, then $\Theta(u)$ gives a ring shift factor in Definition 6.1. Moreover, consider

$$\Theta(1,u)=[u]!_q\,,$$
$$\Theta(2,u)=[u]!_q^2\,,$$
$$\Theta(3,u)=[u]!_q^2\,.$$

Then, Proposition 8.19 and Theorem 15.11 give q-polynomials with positive integer coefficients by convolutions of $\left(\frac{1}{[m]_q}\right)_{m\in\mathbb{Z}_{\geq 0}}$ and $\left(\frac{1}{[m]_q^2}\right)_{m\in\mathbb{Z}_{\geq 0}}$. However, we discuss parcel convolutions with the same weights in this manuscript.

Consider $\sum_{\lambda\in\mathbb{Z}_{\geq 0}}\frac{1}{(\lambda)_q}t^{\lambda}$ and $\sum_{\lambda\in\mathbb{Z}_{\geq 0}}\frac{q^{\frac{\lambda^2}{2}}}{(\lambda)_q}t^{\lambda}$. Then, for Schoenberg's Pólya frequency on real numbers [Scho, Theorem 3] (see [Bra, Edr]), we conjecture the following analog on polynomials. Proposition 8.19 gives the $d=2$ case of the conjecture.

Conjecture 15.13. *Let $h,\lambda,\rho,w\in\mathbb{Z}_{\geq 1}$ and $d\in\mathbb{Z}_{\geq 3}$. Consider $M_{w,h},N_{w,h}\in M_{h,h}(\mathbb{Q}(q^{\frac{1}{2}}))$ such that $M_{w,h,i,j}=\frac{1}{(j-i)_q^w}$ and $N_{w,h,i,j}=\frac{q^{\frac{(j-i)^2}{2}}}{(j-i)_q^w}$ if $0\leq j-i$, and $M_{w,h,i,j}=N_{w,h,i,j}=0$ otherwise. Moreover, for $\alpha,\beta\in\mathrm{T}_<(d,h)$ such that $\beta\geq\alpha$, let*

$$F(w,\rho,\lambda,\alpha,\beta)=\frac{\prod_{i\in[\![d]\!]}(\beta_i-\alpha_1+\lambda)_{q^\rho}^w}{\prod_{i\in[\![d]\!]}(\alpha_i-\alpha_1+\lambda)_{q^\rho}^w}.$$

Then, we have

$$F(w,\rho,\lambda,\alpha,\beta)M_{w,h}(\alpha,\beta)>_q 0,$$
$$F(w,\rho,\lambda,\alpha,\beta)N_{w,h}(\alpha,\beta)>_{q^{\frac{1}{2}}} 0.$$

15.3 Fitting tuples and strictly increasing sequences

We introduce the following notion to discuss fitting tuples and strictly increasing sequences by the extended Cauchy–Binet formula.

Definition 15.14. *Suppose a gate $s\geq 0$ and $l\in\mathbb{Z}_{\geq 1}$. Let $u\in\mathbb{Z}$, $m,n\in\mathbb{Z}^l$, and $k\in\mathbb{Z}^{2l}$. Then, we define*

$$\omega_\nu((s,l,m,n,k),u)=(s,l,\alpha,\beta)$$

such that

$$\alpha=\nu(k)-k_1+u\in\mathbb{Z}^{2l},$$
$$\beta=m\sqcup n+\alpha\in\mathbb{Z}^{2l}.$$

We then obtain strictly increasing sequences from fitting tuples.

Lemma 15.15. *For a fitting $\mu=(s,l,m,n,k)$ and $u\in\mathbb{Z}_{\geq 0}$, let $(s,l,\alpha,\beta)=\omega_\nu(\mu,u)$.*

1. *Then, $(2l,\alpha,\beta)$ is pre-fitting and $\beta\geq\alpha\geq u$.*

2. *If $l = 1$, then $\alpha, \beta \in \mathrm{T}_<(2l, (u, h))$ for some $h \in \mathbb{Z}_{\geq 1}$.*

Proof. Proof of Claim 1. Claim 1 follows from Claim 1 of Proposition 8.10 and Claim 2 of Lemma 3.7.

Proof of Claim 2. Claim 1 gives Claim 2, since α, β are strictly increasing when $l = 1$. □

By the following, we obtain fitting tuples from increasing sequences.

Definition 15.16. *For a gate $s \geq 0$ and $l \in \mathbb{Z}_{\geq 1}$, suppose $\mu = (s, l, \alpha, \beta)$ such that $\alpha, \beta \in \mathbb{Z}^{2l}$. Let $u \in \mathbb{Z}$. Then, we define*

$$\omega_d(\mu, u) = \left(s, l, m_{\alpha,\beta}, n_{\alpha,\beta}, k_{\alpha,\beta,u}\right)$$

such that

$$\begin{aligned} m_{\alpha,\beta} &= (\beta - \alpha)[1 : l] \in \mathbb{Z}^l, \\ n_{\alpha,\beta} &= (\beta - \alpha)[l + 1 : 2l] \in \mathbb{Z}^l, \\ k_{\alpha,\beta,u} &= (u) \mathbin{+\!\!+} (\alpha[2 : 2l] - \alpha[1 : 2l - 1]) \in \mathbb{Z}^{2l}. \end{aligned}$$

When $l = 1$, we define $\omega_d(s, \alpha, \beta, u) = \omega_d(\mu, u)$ for our convenience.

Furthermore, we prove the following reciprocity between ω_ν and ω_d.

Proposition 15.17. *Assume a gate $s \geq 0$ and $l \in \mathbb{Z}_{\geq 1}$. Let $m, n \in \mathbb{Z}^l$, $k \in \mathbb{Z}^{2l}$, and $\mu = (s, l, m, n, k)$. Also, let $\alpha, \beta \in \mathbb{Z}^{2l}$ and $\mu' = (s, l, \alpha, \beta)$. Then, we have the following.*

1. $\omega_d((s, l, \omega_\nu(\mu, \alpha_1)_3, \omega_\nu(\mu, \alpha_1)_4), k_1) = \mu$.
2. $\omega_\nu(\omega_d(\mu', k_1), \alpha_1) = \mu'$.
3. *Suppose $l = 1$, $\alpha_1 \in \mathbb{Z}_{\geq 0}$, $k_1 \in \mathbb{Z}_{\geq 0}$, and $\mu = \omega_d(\mu', k_1)$. Then, μ' satisfies $\alpha - \beta \in [\![s]\!]^{2l}$ and $\alpha, \beta \in \mathrm{T}_<(2l, (\alpha_1, h))$ for some $h \in \mathbb{Z}_{\geq 1}$ if and only if μ is fitting.*

Proof. Proof of Claim 1. Let $\omega_\nu(\mu, \alpha_1) = (s, l, \alpha', \beta')$. Then, we have

$$\begin{aligned} (k_1) \mathbin{+\!\!+} (\alpha'[2 : 2l] - \alpha'[1 : 2l - 1]) &= (k_1) \mathbin{+\!\!+} ((\nu(k) - k_1 + \alpha_1)[2 : 2l] - (\nu(k) - k_1 + \alpha_1)[1 : 2l - 1]) \\ &= (k_1) \mathbin{+\!\!+} (\nu(k)[2 : 2l] - \nu(k)[1 : 2l - 1]) \\ &= k. \end{aligned}$$

Hence, Claim 1 holds by $\beta' - \alpha' = m \mathbin{+\!\!+} n$.

Proof of Claim 2. Let $\omega_d(\mu', k_1) = (s, l, m', n', k')$. Then, $k' = (k_1) \mathbin{+\!\!+} (\alpha[2 : 2l] - \alpha[1 : 2l - 1])$ gives

$$\nu(k') - k_1 + \alpha_1 = (k_1, k_1 + \alpha_2 - \alpha_1, \ldots, k_1 + \alpha_{2l} - \alpha_1) - k_1 + \alpha_1 = \alpha.$$

Thus, Claim 2 follows from $m' \mathbin{+\!\!+} n' = \beta - \alpha$.

Proof of Claim 3. The if part follows from Claim 2 and Claim 2 of Lemma 15.15. We now prove the only if part. First, $\alpha - \beta \in [\![s]\!]^{2l}$ gives the inclusion condition of μ. Second, $k_1 \in \mathbb{Z}_{\geq 0}$ and $\alpha \in \mathrm{T}_<(2l, (\alpha_1, h))$ imply the lower slope condition of μ. Third, $\beta \in \mathrm{T}_<(2l, (\alpha_1, h))$ gives the upper slope condition of μ. □

15.3.1 Merged determinants by Toeplitz matrices and ring shift indices

We consider the following Toeplitz matrices.

Definition 15.18. *For $l = 1$, suppose $\mathscr{F} = (\mathscr{F}_m \in \mathbb{Q}(\mathfrak{X}))_{m \in \mathbb{Z}^l}$. For $h \in \mathbb{Z}_{\geq 1}$ and $i, j \in [\![h]\!]$, we write a matrix $M_{\mathscr{F},h} \in M_{h,h}(\mathbb{Q}(\mathfrak{X}))$ such that its (i, j)-element $M_{\mathscr{F},h,i,j}$ is $\mathscr{F}_{(j-i)}$.*

In particular, we have the following by ω_d.

Lemma 15.19. *Suppose $l = 1$ and $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$. Let $d = 2$. Consider $h \in \mathbb{Z}_{\geq 2}$ and $\alpha,\beta \in \mathrm{T}_<(d,h)$. Then, we have the following.*

1. *$M_{\mathscr{F},h}(\alpha,\beta) \neq 0$ implies $\beta - \alpha \in [\![s]\!]^d$.*
2. *Let $\mathscr{F}$ be $\succ'$-merged-log-concave. Then, $\omega_d(s,\alpha,\beta,u)$ is fitting for each $u \in \mathbb{Z}_{\geq 0}$ if and only if $M_{\mathscr{F},h}(\alpha,\beta) \neq 0$.*

Proof. Proof of Claim 1. Assume $\beta_1 - \alpha_1 < s_1$. Then, $\mathscr{F}_{(\beta_1-\alpha_1)} = 0$. Also, $\mathscr{F}_{(\beta_1-\alpha_2)} = 0$, since $\beta_1 - \alpha_2 < s_1$ by $\alpha_2 > \alpha_1$. Hence, we obtain the contradiction

$$0 \neq M_{\mathscr{F},h}(\alpha,\beta) = \mathscr{F}_{(\beta_1-\alpha_1)}\mathscr{F}_{(\beta_2-\alpha_2)} - \mathscr{F}_{(\beta_2-\alpha_1)}\mathscr{F}_{(\beta_1-\alpha_2)} = 0.$$

The same contradiction occurs when either $\beta_1 - \alpha_1 > s_2$, $\beta_2 - \alpha_2 < s_1$, or $\beta_2 - \alpha_2 > s_2$.

Proof of Claim 2. Let $\omega_d(s,\alpha,\beta,u) = \big(s,l,m_{\alpha,\beta},n_{\alpha,\beta},k_{\alpha,\beta,u}\big)$. First, we prove the if part. Since $k_{\alpha,\beta,u} = (u,\alpha_2-\alpha_1)$, $k_{\alpha,\beta,u} \geq (0,1)$ by $u \geq 0$ and $\alpha_2 > \alpha_1$. Furthermore, since $m_{\alpha,\beta} \mathbin{+\!\!+} n_{\alpha,\beta} = \beta - \alpha$, $\beta_2 > \beta_1$ gives

$$n_{\alpha,\beta} + k_{\alpha,\beta,u,2} = (\beta_2 - \alpha_1) > (\beta_1 - \alpha_1) = m_{\alpha,\beta}.$$

Thus, Lemma 3.10 and Claim 1 give the if part. Second, we prove the only if part. We have

$$\begin{aligned} m_{\alpha,\beta} \boxminus k_{\alpha,\beta,u} &= (\beta_1 - \alpha_1 - (\alpha_2 - \alpha_1)) = (\beta_1 - \alpha_2)\,, \\ n_{\alpha,\beta} \boxplus k_{\alpha,\beta,u} &= (\beta_2 - \alpha_2 + (\alpha_2 - \alpha_1)) = (\beta_2 - \alpha_1)\,. \end{aligned}$$

Then, $M_{\mathscr{F},h}(\alpha,\beta) = \mathscr{F}_{m_{\alpha,\beta}}\mathscr{F}_{n^\vee_{\alpha,\beta}} - \mathscr{F}_{m_{\alpha,\beta}\boxminus k_{\alpha,\beta,u}}\mathscr{F}_{(n_{\alpha,\beta}\boxplus k_{\alpha,\beta,u})^\vee}$. Thus, the only if part holds by the $\succ'$-merged-log-concavity of $\mathscr{F}$. □

Also, we introduce the following notation.

Definition 15.20. *Let $l \in \mathbb{Z}_{\geq 1}$ and $w \in \mathbb{Z}^l_{\geq 0}$. Consider an indeterminate $y \in \mathbb{Q}(\mathfrak{X})^l$ and $\phi(y) \in \prod_{i \in [\![l]\!]} \mathbb{Q}(y_i)$. Then, for each $u \in \mathbb{Z}^l$, we define*

$$\tilde{\Theta}(l,w,y,\phi,u) = \begin{cases} \prod \phi(y)^{u \circ w} \cdot [u]!^w_y & \text{if } u \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

We then realize merged determinants by Toeplitz matrices and ring shift indices.

Proposition 15.21. *Let $l = 1$. Suppose $\mathscr{F}_i = \Lambda(s_i,l,w,\succ_i,f_{i,s_i},\phi,\rho_i,x_i,\mathfrak{X})$ for $i \in [\![3]\!]$ such that $\mathscr{F}_i$ is $\succ'_i$-merged-log-concave for each $i \in [\![2]\!]$. Let*

$$C = (s,l,w,O,\phi,\rho,x,\mathfrak{X},q,\lambda,o)$$

be the convolution index of $(\mathscr{F}_1,\mathscr{F}_2)$ *and* $\mathscr{F}_3=\mathscr{F}_1*\mathscr{F}_2$. *Let* $y=x_3^{\rho_3}$ *and* $\Theta(u)=\tilde{\Theta}(l,w,y,\phi,(u))$ *for each* $u\in\mathbb{Z}$. *Also, for* $d=2$, $h\in\mathbb{Z}_{\geq 2}$, *and* $\alpha,\beta\in \mathrm{T}_<(d,h)$, *let*

$$R=\left(\Theta,d,\iota^3(h),\alpha,\beta,M_{\mathscr{F}_1,h},M_{\mathscr{F}_2,h},\mathbb{Q}(\mathfrak{X})\right).$$

Then, we have the following.

1. *R is a ring shift index.*
2. *For each* $\gamma\in\mathrm{T}_<(d,h)$ *and* $u\geq 0$, *we have*

$$M_{\mathscr{F}_1,h}(\alpha,\gamma,\Theta,u)=\Delta(\mathscr{F}_1)(s_1,l,w,m_{\alpha,\gamma},n_{\alpha,\gamma},k_{\alpha,\gamma,u},\phi,\rho_1,x_1,\mathfrak{X}), \tag{15.3.1}$$
$$M_{\mathscr{F}_2,h}(\gamma,\beta,\Theta,u)=\Delta(\mathscr{F}_2)(s_2,l,w,m_{\gamma,\beta},n_{\gamma,\beta},k_{\gamma,\beta,u},\phi,\rho_2,x_2,\mathfrak{X}), \tag{15.3.2}$$
$$M_{\mathscr{F}_3,h}(\alpha,\beta,\Theta,u)=\Delta(\mathscr{F}_3)(s_3,l,w,m_{\alpha,\beta},n_{\alpha,\beta},k_{\alpha,\beta,u},\phi,\rho_3,x_3,\mathfrak{X}). \tag{15.3.3}$$

Proof. Proof of Claim 1. First, we prove the Θ-nonsingularity. The base positivity of ϕ in $\mathscr{F}_3$ gives $\prod\phi(x_3)^w>_{\mathrm{O}_{\mathfrak{X}}}0$. Thus, $\prod\phi(y)^w\neq 0$ by $\rho_3\in\mathbb{Z}^l_{\geq 1}$. Moreover, $q^{\lambda_1}>_{\mathrm{O}_{\mathfrak{X}}}0$ by Claim 1 of Lemma 5.22. This implies

$$y_1=q^{\lambda_3\rho_{3,1}}=q^{\lambda_1 o_1\rho_{3,1}}>_{\mathrm{O}_{\mathfrak{X}}}0.$$

Hence, we have the Θ-nonsingularity, since $[(u)]!^w_y\neq 0$ for each $u\geq 0$.

Second, let us prove the $[\![3]\!]$- and $[\![2,3]\!]$-product nonsingularities. Assume $\gamma\in\mathrm{T}_<(d,h,\alpha,\beta,M_{\mathscr{F}_1,h},M_{\mathscr{F}_2,h})$. Then, Claim 1 of Lemma 15.19 implies $\gamma_1-\alpha_1\geq 0$ by $M_{\mathscr{F}_1,h}(\alpha,\gamma)M_{\mathscr{F}_2,h}(\gamma,\beta)\neq 0$. Thus, we obtain the $[\![3]\!]$-product nonsingularity by the Θ-nonsingularity and $\gamma_2>\gamma_1$. Also, we have $0=\min(0,\gamma_1-\alpha_1)$, and $\beta_1-\gamma_1\geq 0$ by $M_{\mathscr{F}_1,h}(\alpha,\gamma)M_{\mathscr{F}_2,h}(\gamma,\beta)\neq 0$ and Claim 1 of Lemma 15.19. Thus, the $[\![2,3]\!]$-product nonsingularity follows from the Θ-nonsingularity and $\beta_2>\beta_1$. In particular, Claim 1 holds.

Proof of Claim 2. We prove equation (15.3.1), since equations (15.3.2) and (15.3.3) hold analogously. Let

$$a_{\alpha,\gamma,u}=\nu(k_{\alpha,\gamma,u}),$$
$$b_{\alpha,\gamma,u}=\nu(m_{\alpha,\gamma},n_{\alpha,\gamma},k_{\alpha,\gamma,u}).$$

Then, $k_{\alpha,\gamma,u}=(u,\alpha_2-\alpha_1)$ and $m_{\alpha,\gamma}+\!\!\!+\,n_{\alpha,\gamma}=\gamma-\alpha$ give

$$a_{\alpha,\gamma,u}=(u,\alpha_2-\alpha_1+u), \tag{15.3.4}$$
$$b_{\alpha,\gamma,u}=a_{\alpha,\gamma,u}+m_{\alpha,\gamma}+\!\!\!+\,n_{\alpha,\gamma}=(\gamma_1-\alpha_1+u,\gamma_2-\alpha_1+u), \tag{15.3.5}$$
$$m_{\alpha,\gamma}\boxminus k_{\alpha,\gamma,u}=(\gamma_1-\alpha_1)-(\alpha_2-\alpha_1)=(\gamma_1-\alpha_2), \tag{15.3.6}$$
$$n_{\alpha,\gamma}\boxplus k_{\alpha,\gamma,u}=(\gamma_2-\alpha_2)+(\alpha_2-\alpha_1)=(\gamma_2-\alpha_1). \tag{15.3.7}$$

First, suppose $\gamma_1-\alpha_1+u<0$. Then,

$$\Theta(\alpha,\gamma,u)=\frac{\prod_{i\in[\![d]\!]}\Theta(\gamma_i-\alpha_1+u)}{\prod_{i\in[\![d]\!]}\Theta(\alpha_i-\alpha_1+u)}=0,$$
$$\Upsilon(s_1,l,w,m_{\alpha,\gamma},n_{\alpha,\gamma},k_{\alpha,\gamma,u},\phi,\rho_1,x_1,\mathfrak{X})=0$$

by equation (15.3.5). Hence, equation (15.3.1) holds by $0=0$.

Second, suppose $\gamma_1 - \alpha_1 + u \geq 0$. Then, since $\alpha_2 - \alpha_1 + u \geq 0$ by $u \geq 0$, equations (15.3.4), (15.3.5), (15.3.6), and (15.3.7) give

$$\begin{aligned}
M_{\mathscr{F}_1,h}(\alpha,\gamma,\Theta,u) &= \Theta(\alpha,\gamma,u) M_{\mathscr{F}_1,h}(\alpha,\gamma) \\
&= \frac{\Theta(\gamma_2-\alpha_1+u)\Theta(\gamma_1-\alpha_1+u)}{\Theta(\alpha_2-\alpha_1+u)\Theta(u)} \cdot (M_{\mathscr{F}_1,h,\alpha_1,\gamma_1} M_{\mathscr{F}_1,h,\alpha_2,\gamma_2} - M_{\mathscr{F}_1,h,\alpha_1,\gamma_2} M_{\mathscr{F}_1,h,\alpha_2,\gamma_1}) \\
&= \frac{\prod(\phi(y)^{\sqcup})^{(\gamma_2-\alpha_1+u,\gamma_1-\alpha_1+u)\circ w^{\sqcup}} [(\gamma_2-\alpha_1+u)]!^{w}_{y} [(\gamma_1-\alpha_1+u)]!^{w}_{y}}{\prod(\phi(y)^{\sqcup})^{(\alpha_2-\alpha_1+u,u)\circ w^{\sqcup}} [(\alpha_2-\alpha_1+u)]!^{w}_{y} [(u)]!^{w}_{y}} \\
&\quad \cdot (\mathscr{F}_{1,(\gamma_1-\alpha_1)} \mathscr{F}_{1,(\gamma_2-\alpha_2)} - \mathscr{F}_{1,(\gamma_2-\alpha_1)} \mathscr{F}_{1,(\gamma_1-\alpha_2)}) \\
&= \frac{\prod(\phi(y)^{\sqcup})^{b_{\alpha,\gamma,u}\circ w^{\sqcup}} [b_{\alpha,\gamma,u}]!^{w^{\sqcup}}_{y^{\sqcup}}}{\prod(\phi(y)^{\sqcup})^{a_{\alpha,\gamma,u}\circ w^{\sqcup}} [a_{\alpha,\gamma,u}]!^{w^{\sqcup}}_{y^{\sqcup}}} \cdot (\mathscr{F}_{1,m_{\alpha,\gamma}} \mathscr{F}_{1,n^{\vee}_{\alpha,\gamma}} - \mathscr{F}_{1,m_{\alpha,\gamma} \boxminus k_{\alpha,\gamma,u}} \mathscr{F}_{1,(n_{\alpha,\gamma} \boxplus k_{\alpha,\gamma,u})^{\vee}}).
\end{aligned}$$

Furthermore, we have

$$\frac{\prod(\phi(y)^{\sqcup})^{b_{\alpha,\gamma,u}\circ w^{\sqcup}} [b_{\alpha,\gamma,u}]!^{w^{\sqcup}}_{y^{\sqcup}}}{\prod(\phi(y)^{\sqcup})^{a_{\alpha,\gamma,u}\circ w^{\sqcup}} [a_{\alpha,\gamma,u}]!^{w^{\sqcup}}_{y^{\sqcup}}} = \Upsilon(s_1, l, w, m_{\alpha,\gamma}, n_{\alpha,\gamma}, k_{\alpha,\gamma,u}, \phi, \rho_1, x_1, \mathfrak{X}),$$

since $y = x_3^{\rho_3} = \left(q^{\lambda_3}\right)^{o_1^{-1}\rho_1} = \left(q^{\lambda_1 o_1 o_1^{-1} \rho_{1,1}}\right) = \left(q^{\lambda_1 \rho_{1,1}}\right) = x_1^{\rho_1}$ in the convolution index C. Thus, we obtain equation (15.3.1). □

15.4 Merged-log-concavity

If there is a squaring order $\succ'_3$ compatible to $\succ'_1, \succ'_2$, and $\succ_3$ in Proposition 15.21, then Theorem 15.11 and Proposition 15.21 give the $\succeq'_3$-merged-log-concavity of $\mathscr{F}_3$. But, we need the $\succ'_3$-merged-log-concavity of $\mathscr{F}_3$ for polynomials with positive integer coefficients. Hence, we introduce the following notation to obtain fitting tuples for $M_{\mathscr{F}_1,h}(\alpha,\gamma,\Theta,u)$ and $M_{\mathscr{F}_2,h}(\gamma,\beta,\Theta,u)$ in Proposition 15.21.

Definition 15.22. *Suppose gates $s_1, s_2 \geq 0$. For each $u \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{Z}^2$, we define*

$$\mathrm{FT}_{<}(s_1,s_2,\alpha,\beta,u) = \{\gamma \in \mathrm{T}_{<}(2,(\alpha_1,\beta_2)) \mid \omega_d(s_1,\alpha,\gamma,u) \text{ and } \omega_d(s_2,\gamma,\beta,u) \text{ are fitting.}\}.$$

Then, we prove the following existence of fitting tuples.

Lemma 15.23. *Let $l = 1$ and $\alpha_1 \in \mathbb{Z}_{\geq 1}$. Assume the following:*

1. $s_3 = s_1 + s_2$ *for gates* $s_1, s_2, s_3 \geq 0$*;*
2. *a fitting* $\mu = (s_3, l, m, n, k)$*;*
3. $(s_3, l, \alpha, \beta) = \omega_\nu(\mu, \alpha_1)$*;*
4. $\gamma = (\max(s_{1,1}+\alpha_1, \beta_1 - s_{2,2}), \min(s_{1,2}+\alpha_2, \beta_2 - s_{2,1})) \in \mathbb{Z}^2$.

Then, we have the following.

(a) There exist the following inequalities:

$$\gamma_1 \leq \min(s_{1,2}+\alpha_1, \beta_1 - s_{2,1}); \tag{15.4.1}$$

$$\max(s_{1,1}+\alpha_2, \beta_2 - s_{2,2}) \leq \gamma_2; \tag{15.4.2}$$

$$\gamma_1 < \gamma_2. \tag{15.4.3}$$

(b) For each $u \in \mathbb{Z}_{\geq 0}$, there exists $\gamma \in \mathrm{FT}_<(s_1, s_2, \alpha, \beta, u)$.

Proof. We state the following inequalities to prove Claims (a) and (b). First, the inclusion condition of Assumption 2 and $\beta - \alpha = m \mathbin{+\!\!+} n$ of Assumption 3 imply

$$s_{3,1} \leq \beta_1 - \alpha_1 \leq s_{3,2}, \tag{15.4.4}$$

$$s_{3,1} \leq \beta_2 - \alpha_2 \leq s_{3,2}. \tag{15.4.5}$$

Second, Assumption 2 and Claim 2 of Lemma 15.15 give some $h \in \mathbb{Z}_{\geq 1}$ such that

$$1 \leq \alpha_1 < \alpha_2 \leq h, \tag{15.4.6}$$

$$1 \leq \beta_1 < \beta_2 \leq h. \tag{15.4.7}$$

Proof of Claim (a). First, we prove inequality (15.4.1). Then, by Assumption 1 and inequality (15.4.4), we have

$$(s_{1,2} + \alpha_1) - (\beta_1 - s_{2,2}) = s_{1,2} + s_{2,2} + \alpha_1 - \beta_1 = s_{3,2} - (\beta_1 - \alpha_1) \geq 0,$$
$$(\beta_1 - s_{2,1}) - (s_{1,1} + \alpha_1) = \beta_1 - \alpha_1 - (s_{1,1} + s_{2,1}) = \beta_1 - \alpha_1 - s_{3,1} \geq 0.$$

Therefore, we obtain inequality (15.4.1), because Assumption 1 implies

$$(s_{1,2} + \alpha_1) - (s_{1,1} + \alpha_1) = s_{1,2} - s_{1,1} \geq 0,$$
$$(\beta_1 - s_{2,1}) - (\beta_1 - s_{2,2}) = s_{2,2} - s_{2,1} \geq 0.$$

Second, inequality (15.4.2) holds similarly by Assumption 1 and inequality (15.4.5).

Third, we prove inequality (15.4.3). By Assumption 1 and inequalities (15.4.6) and (15.4.7), we have

$$(s_{1,2} + \alpha_2) - (s_{1,1} + \alpha_1) = (s_{1,2} - s_{1,1}) + (\alpha_2 - \alpha_1) > 0,$$
$$(\beta_2 - s_{2,1}) - (\beta_1 - s_{2,2}) = (s_{2,2} - s_{2,1}) + (\beta_2 - \beta_1) > 0.$$

Furthermore, Assumption 1 and inequalities (15.4.4) and (15.4.7) give

$$(\beta_2 - s_{2,1}) - (s_{1,1} + \alpha_1) = \beta_2 - \alpha_1 - s_{3,1} > \beta_1 - \alpha_1 - s_{3,1} \geq 0.$$

If $s_{3,2} = \infty$, then $\gamma_1 = s_{1,1} + \alpha_1$ if $s_{2,2} = \infty$ or $\gamma_2 = \beta_2 - s_{2,1}$ if $s_{1,2} = \infty$. Thus, suppose $s_{3,2} < \infty$. Then, Assumption 1 and inequalities (15.4.4) and (15.4.6) imply

$$(s_{1,2} + \alpha_2) - (\beta_1 - s_{2,2}) = s_{3,2} - \beta_1 + \alpha_2 > s_{3,2} - \beta_1 + \alpha_1 = s_{3,2} - (\beta_1 - \alpha_1) \geq 0.$$

Thus, inequality (15.4.3) follows.

Proof of Claim (b). First, we prove

$$\gamma \in \mathrm{T}_<(2, (\alpha_1, \beta_2)). \tag{15.4.8}$$

Since $s_{1,1}, s_{2,1} \geq 0$ by Assumption 1, Assumption 4 implies $\alpha_1 \leq \gamma_1$ and $\gamma_2 \leq \beta_2$. Therefore, inclusion (15.4.8) holds by inequality (15.4.3).

Second, we confirm that $\omega_d(s_1, \alpha, \gamma, u) = \left(s_1, l, m_{\alpha,\gamma}, n_{\alpha,\gamma}, k_{\alpha,\gamma,u}\right)$ is fitting. By Assumption 4 and inequalities (15.4.1) and (15.4.2), each $i \in [\![2]\!]$ gives inequalities $s_{1,1} + \alpha_i \leq \gamma_i \leq s_{1,2} + \alpha_i$ and $\beta_i - s_{2,2} \leq \gamma_i \leq \beta_i - s_{2,1}$, which are equivalent to

$$s_{1,1} \leq \gamma_i - \alpha_i \leq s_{1,2}, \tag{15.4.9}$$

$$s_{2,1} \leq \beta_i - \gamma_i \leq s_{2,2}. \tag{15.4.10}$$

Then, inequality (15.4.9) implies

$$m_{\alpha,\gamma} \mathbin{+\!\!+} n_{\alpha,\gamma} = (\gamma_i - \alpha_i)_{i \in [\![2]\!]} \in [\![s_1]\!]^{2l}.$$

Moreover, inequality (15.4.3) gives

$$n_{\alpha,\gamma,1} + k_{\alpha,\gamma,u,2} = \gamma_2 - \alpha_1 > \gamma_1 - \alpha_1 = m_{\alpha,\gamma,1}.$$

Hence, $\omega_d(s_1,\alpha,\gamma,u)$ is fitting by inequality (15.4.6) and Lemma 3.10.

Third, we prove that $\omega_d(s_2,\gamma,\beta,u) = \left(s_2,l,m_{\gamma,\beta},n_{\gamma,\beta},k_{\gamma,\beta,u}\right)$ is fitting. By inequality (15.4.10), we have

$$m_{\gamma,\beta} \mathbin{+\!\!+} n_{\gamma,\beta} = (\beta_i - \gamma_i)_{i \in [\![2]\!]} \in [\![s_2]\!]^{2l}.$$

Also, inequality (15.4.7) gives

$$n_{\gamma,\beta,1} + k_{\gamma,\beta,u,2} = \beta_2 - \gamma_1 > \beta_1 - \gamma_1 = m_{\gamma,\beta,1}.$$

Therefore, $\omega_d(s_2,\gamma,\beta,u)$ is fitting by inequality (15.4.3) and Lemma 3.10. □

Moreover, we rewrite $\mathrm{FT}_<(s_1,s_2,\alpha,\beta,u)$ as follows.

Lemma 15.24. *Let $\alpha,\beta \in \mathrm{T}_<(2,h)$ for some $h \in \mathbb{Z}_{\geq 1}$. Suppose $u \in \mathbb{Z}_{\geq 0}$. Then, $\gamma \in \mathrm{FT}_<(s_1,s_2,\alpha,\beta,u)$ if and only if $\gamma \in \mathrm{T}_<(2,h)$ and both $\omega_d(s_1,\alpha,\gamma,u)$ and $\omega_d(s_2,\gamma,\beta,u)$ are fitting.*

Proof. The only if part is clear. Hence, we prove the if part. The fitting $\omega_d(s_1,\alpha,\gamma,u)$ and $\omega_d(s_2,\gamma,\beta,u)$ give $\alpha_1 \leq \gamma_1$ and $\gamma_2 \leq \beta_2$, because

$$\begin{aligned} m_{\alpha,\gamma} &= (\gamma_1 - \alpha_1) \in [\![s_1]\!]^l, \\ n_{\gamma,\beta} &= (\beta_2 - \gamma_2) \in [\![s_2]\!]^l \end{aligned}$$

by $s_1,s_2 \geq 0$. Thus, $\gamma \in \mathrm{T}_<(2,(\alpha_1,\beta_2))$, since $\gamma_1 < \gamma_2$ by $\gamma \in \mathrm{T}_<(2,h)$. □

We now state the following merged-log-concavity of parcel convolutions. Its weight-zero case applies to the convolutions of strongly $>_q$-log-concave polynomials, which carry convolution indices by Claim 3c of Lemma 15.2.

Theorem 15.25. *Let $l = 1$. Consider $\mathscr{F}_i = \Lambda(s_i,l,w,\succ_i,f_{i,s_i},\phi,\rho_i,x_i,\mathfrak{X})$ for $i \in [\![3]\!]$ such that $\mathscr{F}_3 = \mathscr{F}_1 * \mathscr{F}_2$. For a fitting $\mu = (s_3,l,m,n,k)$ and $\alpha_1 \in \mathbb{Z}_{\geq 1}$, let $(s_3,l,\alpha,\beta) = \omega_\nu(\mu,\alpha_1)$. Then, we obtain*

$$\begin{aligned} &\Delta(\mathscr{F}_3)(s_3,l,w,m,n,k,\phi,\rho_3,x_3,\mathfrak{X}) \\ &= \sum_{\gamma \in \mathrm{FT}_<(s_1,s_2,\alpha,\beta,k_1)} \Delta(\mathscr{F}_1)(s_1,l,w,m_{\alpha,\gamma},n_{\alpha,\gamma},k_{\alpha,\gamma,k_1},\phi,\rho_1,x_1,\mathfrak{X}) \\ &\cdot \Delta(\mathscr{F}_2)(s_2,l,w,m_{\gamma,\beta},n_{\gamma,\beta},k_{\gamma,\beta,\gamma_1-\alpha_1+k_1},\phi,\rho_2,x_2,\mathfrak{X}). \end{aligned} \tag{15.4.11}$$

Moreover, let $\mathscr{F}_i$ be $\succ'_i$-merged-log-concave for $i \in [\![2]\!]$ with $O_i = \{\succeq_i,\succ_i\}$ for $i \in [\![3]\!]$ and $O'_i = \{\succeq'_i,\succ'_i\}$ for $i \in [\![2]\!]$. If there are squaring orders $O'_3 = \left\{\succeq'_3,\succ'_3\right\} \supseteq O'_1,O'_2,O_3$, then $\mathscr{F}_3$ is $\succ'_3$-merged-log-concave by

$$\Delta(\mathscr{F}_3)(s_3,l,w,m,n,k,\phi,\rho_3,x_3,\mathfrak{X}) \succ'_3 0. \tag{15.4.12}$$

Proof. First, we prove equation (15.4.11). Consider the convolution index $C=(s,l,w,O,\phi,\rho,x,\mathfrak{X},q,\lambda,o)$ of $(\mathscr{F}_1,\mathscr{F}_2)$. Also, let $y=x_3^{\rho_3}$ and $\Theta(u)=\tilde{\Theta}(l,w,y,\phi,(u))$ for each $u\in\mathbb{Z}$. Then, we have a ring shift index $R=\left(\Theta,d,\iota^3(h),\alpha,\beta,M_{\mathscr{F}_1,h},M_{\mathscr{F}_2,h},\mathbb{Q}(\mathfrak{X})\right)$ for some $h\in\mathbb{Z}_{\geq 1}$ by Claim 1 of Proposition 15.21.

We then have

$$M_{\mathscr{F}_3,h}(\alpha,\beta,\Theta,k_1)=\Delta(\mathscr{F}_3)(s_3,l,w,m,n,k,\phi,\rho_3,x_3,\mathfrak{X})$$

by the convolution index C and Claim 2 of Proposition 15.21, because $\omega_d(s_3,\alpha,\beta,k_1)=(s_3,l,m,n,k)$ by Claim 1 of Proposition 15.17. Furthermore, if $M_{\mathscr{F}_1,h}(\alpha,\gamma)M_{\mathscr{F}_2,h}(\gamma,\beta)\neq 0$, then Claim 1 of Lemma 15.19 implies $\gamma_1-\alpha_1\geq 0$. In particular, Claim 2 of Lemma 15.19 implies

$$\begin{aligned}M_{\mathscr{F}_1,h}(\alpha,\gamma,\Theta,k_1)&=\Delta(\mathscr{F}_1)(s_1,l,w,m_{\alpha,\gamma},n_{\alpha,\gamma},k_{\alpha,\gamma,k_1},\phi,\rho_1,x_1,\mathfrak{X}),\\ M_{\mathscr{F}_2,h}(\gamma,\beta,\Theta,\gamma_1-\alpha_1+k_1)&=\Delta(\mathscr{F}_2)(s_2,l,w,m_{\gamma,\beta},n_{\gamma,\beta},k_{\gamma,\beta,\gamma_1-\alpha_1+k_1},\phi,\rho_2,x_2,\mathfrak{X}).\end{aligned}$$

We have $M_{\mathscr{F}_3,h}=M_{\mathscr{F}_1,h}M_{\mathscr{F}_2,h}$, since $\mathscr{F}_{3,m_1}=\sum_{m_2\in\mathbb{Z}^l}\mathscr{F}_{1,m_2}\cdot\mathscr{F}_{2,m_1-m_2}$ for each $m_1\in\mathbb{Z}^l$. Hence, Theorem 15.11 with the ring shift index R gives

$$\begin{aligned}\Delta(\mathscr{F}_3)(s_3,l,w,m,n,k,\phi,\rho_3,x_3,\mathfrak{X})&=M_{\mathscr{F}_3,h}(\alpha,\beta,\Theta,k_1)\\ &=\sum_{\gamma\in\mathrm{T}_<(2,h,\alpha,\beta,M_{\mathscr{F}_1,h},M_{\mathscr{F}_2,h})}M_{\mathscr{F}_1,h}(\alpha,\gamma,\Theta,k_1)\cdot M_{\mathscr{F}_2,h}(\gamma,\beta,\Theta,\gamma_1-\alpha_1+k_1)\\ &=\sum_{\gamma\in\mathrm{T}_<(2,h,\alpha,\beta,M_{\mathscr{F}_1,h},M_{\mathscr{F}_2,h})}\Delta(\mathscr{F}_1)(s_1,l,w,m_{\alpha,\gamma},n_{\alpha,\gamma},k_{\alpha,\gamma,k_1},\phi,\rho_1,x_1,\mathfrak{X})\\ &\quad\cdot\Delta(\mathscr{F}_2)(s_2,l,w,m_{\gamma,\beta},n_{\gamma,\beta},k_{\gamma,\beta,\gamma_1-\alpha_1+k_1},\phi,\rho_2,x_2,\mathfrak{X}).\end{aligned}$$

This implies equation (15.4.11) by Claim 2 of Lemma 15.19 and Lemma 15.24.

Second, we prove inequality (15.4.12). Let

$$\lambda=(\max(s_{1,1}+\alpha_1,\beta_1-s_{2,2}),\min(s_{1,2}+\alpha_2,\beta_2-s_{2,1}))\in\mathbb{Z}^2.$$

Then, $\lambda\in\mathrm{FT}_<(s_1,s_2,\alpha,\beta,k_1)$ by Claim (b) of Lemma 15.23. Also, the merged-log-concavity of $\mathscr{F}_1$ and $\mathscr{F}_2$ says

$$\begin{aligned}\Delta(\mathscr{F}_1)(s_1,l,w,m_{\alpha,\lambda},n_{\alpha,\lambda},k_{\alpha,\lambda,k_1},\phi,\rho_1,x_1,\mathfrak{X})&\succ'_1 0,\\ \Delta(\mathscr{F}_2)(s_2,l,w,m_{\lambda,\beta},n_{\lambda,\beta},k_{\lambda,\beta,\lambda_1-\alpha_1+k_1},\phi,\rho_2,x_2,\mathfrak{X})&\succ'_2 0.\end{aligned}$$

Thus, equation (15.4.11) and the compatibility $O'_3\Supset O'_1,O'_2$ imply inequality (15.4.12). □

Example 15.26. We explicitly compute both sides of equation (15.4.11) in Theorem 15.25. Let $l=1$, $s_1=s_2=s_3=(0,\infty)$, $w=(1)$, $x_1=x_2=x_3=(q)$, $\mathfrak{X}=\{q\}$, $\succ_1=\succ_2=\succ_3=>_q$, and $\gamma=((0,0,0))$. Then, for $i\in[\![2]\!]$, suppose parcels

$$\mathscr{F}_i=\Lambda(s_i,l,w,\succ_i,\Psi_{s,\gamma,q},x_i,\mathfrak{X}).$$

Moreover, let $\rho_1=\rho_2=\rho_3=(1)$, $\lambda=\iota^3(1)$, and $o=\iota^2(1)$. Then, Claims 3a and 3b of Lemma 15.2 give the convolution index $C=(s,l,w,O,\phi,\rho,x,\mathfrak{X},q,\lambda,o)$. Thus, consider

$$\mathscr{F}_3=\mathscr{F}_1*\mathscr{F}_2=\Lambda(s_3,l,w,\succ_3,f,x_3,\mathfrak{X})$$

such that $f_m = \prod \phi(x_3)^{m \circ w} \cdot [m]!^{w}_{x_3} \cdot \sum_{i+j=m} \mathscr{F}_{1,i}\mathscr{F}_{2,j}$ for each $m \in [\![s]\!]^l$.

Furthermore, Lemma 3.10 gives a fitting $\mu = (s_3, l, m, n, k)$ such that $m = n = (1)$ and $k = (0,1)$. Then, $\alpha = (1,2)$ and $\beta = (2,3)$ satisfy $(s_3, l, \alpha, \beta) = \omega_\vee(\mu, \alpha_1)$. Thus, $\mathrm{FT}_<(s_1, s_2, \alpha, \beta, k_1)$ consists of $\gamma_1 = (1,2)$, $\gamma_2 = (1,3)$, and $\gamma_3 = (2,3)$. In particular,

$$\gamma_2 = (\max(s_{1,1} + \alpha_1, \beta_1 - s_{2,2}), \min(s_{1,2} + \alpha_2, \beta_2 - s_{2,1}))$$

as in Claim 4 of Lemma 15.23.

The right-hand side of equation (15.4.11) in Theorem 15.25 sums the following q-polynomials:

$$\begin{aligned}
&\Delta(\mathscr{F}_1)(s_1, l, w, m_{\alpha,\gamma_1}, n_{\alpha,\gamma_1}, k_{\alpha,\gamma_1,k_1}, x_1, \mathfrak{X})\\
&\cdot \Delta(\mathscr{F}_2)(s_2, l, w, m_{\gamma_1,\beta}, n_{\gamma_1,\beta}, k_{\gamma_1,\beta,\gamma_{1,1}-\alpha_1+k_1}, x_2, \mathfrak{X})\\
&= \Delta(\mathscr{F}_1)(s_1, l, w, (0), (0), (0,1), x_1, \mathfrak{X}) \cdot \Delta(\mathscr{F}_2)(s_2, l, w, (1), (1), (0,1), x_2, \mathfrak{X})\\
&= q;\\
&\Delta(\mathscr{F}_1)(s_1, l, w, m_{\alpha,\gamma_2}, n_{\alpha,\gamma_2}, k_{\alpha,\gamma_2,k_1}, x_1, \mathfrak{X})\\
&\cdot \Delta(\mathscr{F}_2)(s_2, l, w, m_{\gamma_2,\beta}, n_{\gamma_2,\beta}, k_{\gamma_2,\beta,\gamma_{2,1}-\alpha_1+k_1}, x_2, \mathfrak{X})\\
&= \Delta(\mathscr{F}_1)(s_1, l, w, (0), (1), (0,1), x_1, \mathfrak{X}) \cdot \Delta(\mathscr{F}_2)(s_2, l, w, (1), (0), (0,2), x_2, \mathfrak{X})\\
&= q+1;\\
&\Delta(\mathscr{F}_1)(s_1, l, w, m_{\alpha,\gamma_3}, n_{\alpha,\gamma_3}, k_{\alpha,\gamma_3,k_1}, x_1, \mathfrak{X})\\
&\cdot \Delta(\mathscr{F}_2)(s_2, l, w, m_{\gamma_3,\beta}, n_{\gamma_3,\beta}, k_{\gamma_3,\beta,\gamma_{3,1}-\alpha_1+k_1}, x_2, \mathfrak{X})\\
&= \Delta(\mathscr{F}_1)(s_1, l, w, (1), (1), (0,1), x_1, \mathfrak{X}) \cdot \Delta(\mathscr{F}_2)(s_2, l, w, (0), (0), (1,1), x_2, \mathfrak{X})\\
&= q.
\end{aligned}$$

Then, we obtain $3q+1$. Therefore, this coincides with the left-hand side:

$$\Delta(\mathscr{F}_3)(s_3, l, w, m, n, k, x_3, \mathfrak{X}) = \frac{(1)_q (2)_q}{(0)_q (1)_q} \left(\mathscr{F}^2_{3,(1)} - \mathscr{F}_{3,(0)}\mathscr{F}_{3,(2)}\right) = (2)_q \left(\left(\frac{2}{(1-q)}\right)^2 - \frac{q+3}{(1-q)(1-q^2)}\right).$$

The following gives polynomials with positive integer coefficients by Theorem 15.25.

Corollary 15.27. *Let $l = 1$. Consider $\mathscr{F}_i = \Lambda(s_i, l, w, \succ_i, f_{i,s_i}, \phi, \rho_i, x_i, \mathfrak{X})$ for $i \in [\![3]\!]$ with the parcel convolution $\mathscr{F}_3 = \mathscr{F}_1 * \mathscr{F}_2$. Also, for each $i \in [\![2]\!]$ and fitting (s_i, l, m, n, k), suppose $\kappa_1 \in \mathbb{Q}_{>0}$ and $\kappa_2 \in \mathbb{Q}$ such that*

$$q^{\kappa_2(m_1+n_1)} \Delta(\mathscr{F}_i)(s_i, l, w, m, n, k, \phi, \rho_i, x_i, \mathfrak{X}) >_{q^{\kappa_1}} 0.$$

Then, for each fitting (s_3, l, m, n, k), we have

$$q^{\kappa_2(m_1+n_1)} \Delta(\mathscr{F}_3)(s_3, l, w, m, n, k, \phi, \rho_3, x_3, \mathfrak{X}) >_{q^{\kappa_1}} 0.$$

Proof. In equation (15.4.11) of Theorem 15.25, we have

$$\begin{aligned}
m + n &= \beta - \alpha,\\
m_{\alpha,\gamma} + n_{\alpha,\gamma} &= \gamma - \alpha,\\
m_{\gamma,\beta} + n_{\gamma,\beta} &= \beta - \gamma.
\end{aligned}$$

Thus, the assertion follows, because

$$\kappa_2(m_{\alpha,\gamma,1} + n_{\alpha,\gamma,1}) + \kappa_2(m_{\gamma,\beta,1} + n_{\gamma,\beta,1}) = \kappa_2(\beta_1 - \alpha_1 + \beta_2 - \alpha_2) = \kappa_2(m_1 + n_1).$$

□

16 Explicit parcel convolutions, critical points, phase transitions, and merged determinants

We compute explicit parcel convolutions of finite gates in this section, as ones of infinite gates appear later.

16.1 A parcel convolution of weight one

Let $s_1 = (1,4)$, $l = 1$, $w = (1)$, $\succ = >_q$, $\rho = (1)$, $x = (q)$, and $\mathfrak{X} = \{q\}$. Then, we have q-Stirling and monomial parcels $\mathscr{F}_1$ and $\mathscr{F}_2$ such that

$$\mathscr{F}_1 = \Lambda(s_1, l, \succ, c_{s_1,l,q}, \mathfrak{X}),$$
$$\mathscr{F}_2 = \Lambda(s_1, l, w, \succ, \Psi_{s_1,((0,1,0)),q}, \rho, x, \mathfrak{X}).$$

Then, Claim 1 of Corollary 10.4 gives the $\succ$-merged-log-concave parcel

$$\mathscr{F}_3 = \Lambda(s_1, l, w, \succ, f_{3,s_1}, \rho, x, \mathfrak{X})$$

such that $f_{3,s_1,m} = q^{m_1} c_{s_1,l,q,m}$ for $m \in [\![s_1]\!]^l$.

Now, let $s_2 = 2s_1 = (2,8)$. Then, Claims 3a and 4b of Lemma 15.2 and Theorem 15.25 provides the $\succ$-merged-log-concave parcel

$$\mathscr{F}_4 = \mathscr{F}_3 * \mathscr{F}_3 = \Lambda(s_2, l, w, \succ, f_{4,s_2}, \rho, x, \mathfrak{X}).$$

Explicitly, we have the following q-polynomials:

$$f_{4,s_2,(2)} = q^9 + 5q^8 + 12q^7 + 18q^6 + 18q^5 + 12q^4 + 5q^3 + q^2; \tag{16.1.1}$$

$$\begin{aligned} f_{4,s_2,(3)} &= 2q^{11} + 12q^{10} + 36q^9 + 70q^8 + 94q^7 + 90q^6 \\ &\quad + 60q^5 + 26q^4 + 6q^3; \end{aligned} \tag{16.1.2}$$

$$f_{4,s_2,(4)} = q^{14} + 7q^{13} + 27q^{12} + 70q^{11} + 133q^{10} + 191q^9 + 212q^8 + 183q^7 + 120q^6 + 55q^5 + 15q^4;$$

$$f_{4,s_2,(5)} = 2q^{16} + 12q^{15} + 40q^{14} + 90q^{13} + 154q^{12} + 214q^{11} + 244q^{10} + 236q^9 + 186q^8 + 122q^7 + 60q^6 + 20q^5;$$

$$\begin{aligned} f_{4,s_2,(6)} &= q^{19} + 5q^{18} + 18q^{17} + 41q^{16} + 74q^{15} + 108q^{14} + 139q^{13} \\ &\quad + 157q^{12} + 154q^{11} + 136q^{10} + 101q^9 + 66q^8 + 35q^7 + 15q^6; \end{aligned}$$

$$\begin{aligned} f_{4,s_2,(7)} &= 2q^{21} + 6q^{20} + 14q^{19} + 20q^{18} + 32q^{17} + 42q^{16} + 50q^{15} + 52q^{14} \\ &\quad + 54q^{13} + 46q^{12} + 40q^{11} + 28q^{10} + 18q^9 + 10q^8 + 6q^7; \end{aligned} \tag{16.1.3}$$

$$\begin{aligned} f_{4,s_2,(8)} &= q^{24} + q^{23} + 2q^{22} + 3q^{21} + 5q^{20} + 5q^{19} + 7q^{18} + 7q^{17} + 8q^{16} \\ &\quad + 7q^{15} + 7q^{14} + 5q^{13} + 5q^{12} + 3q^{11} + 2q^{10} + q^9 + q^8. \end{aligned} \tag{16.1.4}$$

16.1.1 On critical points and phase transitions

For $\theta = (3,7)$, let $P = r_{(1,\infty),\theta}(\pi(1)) = ((s_1, l, m_i, n_i, k_i))_{i \in [\![\theta]\!]}$ such that $m_i = n_i = (i)$ and $k_i = (0,1)$. Then, we have the ideal merged pair $\zeta = (P, \mathscr{F}_4)$. First, ζ has the unique front critical point $\mathrm{F_C}(\zeta) = 0.181093\ldots$ that solves

$$\mathscr{F}_{4,(2)}(q) = \mathscr{F}_{4,(3)}(q)$$

by equations (16.1.1) and (16.1.2). If $\mathrm{Bot}_1(q) = (2)_q$, $\mathrm{Bot}_2(q) = (3)_q$, $\mathrm{Top}_1(q) = f_{4,s_2,(2)}(q)$, and $\mathrm{Top}_2(q) = f_{4,s_2,(3)}(q)$, then we obtain the front phase transition of ζ at $\mathrm{F_C}(\zeta)$ by Lemma 13.25. Second, ζ has the unique rear critical point $\mathrm{R_C}(\mu) = 0.978644\ldots$ that solves

$$\mathscr{F}_{4,(7)}(q) = \mathscr{F}_{4,(8)}(q)$$

by equations (16.1.3) and (16.1.4). Then, Lemma 13.25 gives the rear phase transition of ζ at $\mathrm{R_C}(\zeta)$. Third, ζ has no asymptotic critical points for the finite gate θ.

16.1.2 Polynomials with positive integer coefficients of an ideal merged pair

Let $m_2 = n_2 = (2)$ and $k_2 = (0,1)$. Then, the following q-polynomials are merged determinants of ζ:

$$\begin{aligned}
\Delta(\mathscr{F}_4)(s_2,l,w,m_2,n_2,k_2,\rho,x,\mathfrak{X}) &= q^{20} + 11q^{19} + 60q^{18} + 215q^{17} + 565q^{16} + 1152q^{15} \\
&\quad + 1882q^{14} + 2510q^{13} + 2760q^{12} + 2510q^{11} + 1882q^{10} \\
&\quad + 1152q^{9} + 565q^{8} + 215q^{7} + 60q^{6} + 11q^{5} + q^{4}; \\
\Delta(\mathscr{F}_4)(s_2,l,w,m_3,n_3,k_3,\rho,x,\mathfrak{X}) &= 3q^{25} + 39q^{24} + 253q^{23} + 1091q^{22} + 3500q^{21} + 8862q^{20} + 18351q^{19} \\
&\quad + 31793q^{18} + 46772q^{17} + 58972q^{16} + 64038q^{15} + 59954q^{14} + 48259q^{13} \\
&\quad + 33163q^{12} + 19215q^{11} + 9201q^{10} + 3526q^{9} + 1024q^{8} + 203q^{7} + 21q^{6}; \\
\Delta(\mathscr{F}_4)(s_2,l,w,m_4,n_4,k_4,\rho,x,\mathfrak{X}) &= q^{32} + 15q^{31} + 114q^{30} + 584q^{29} + 2263q^{28} + 7054q^{27} \\
&\quad + 18368q^{26} + 40957q^{25} + 79554q^{24} + 136253q^{23} + 207590q^{22} \\
&\quad + 283136q^{21} + 347192q^{20} + 383691q^{19} + 382378q^{18} \\
&\quad + 343199q^{17} + 276483q^{16} + 198741q^{15} + 126308q^{14} + 70013q^{13} \\
&\quad + 33168q^{12} + 13010q^{11} + 4008q^{10} + 875q^{9} + 105q^{8}.
\end{aligned}$$

In particular, $\Delta(\mathscr{F}_4)(s_2,l,w,m_i,n_i,k_i,\rho,x,\mathfrak{X})$ is a log-concave q-polynomial for each $i \in [\![e(\theta)]\!]$. Also, it is palindromic for each $i = 2,8$, but not for $i \in [\![\theta]\!]$.

16.2 A parcel convolution of weight two

Let $s_1 = (0,2)$, $s_2 = (1,2)$, $l = 1$, $w = (2)$, $\succ = >_q$, $\rho = (1)$, $x = (q)$, and $\mathfrak{X} = \{q\}$. First, we have the q-number parcel

$$\mathscr{F}_1 = \Lambda(s_2,l,\succ,\chi_{s_2,q},\mathfrak{X})$$

and monomial parcels

$$\begin{aligned}
\mathscr{F}_2 &= \Lambda(s_2,l,w,\succ,\Psi_{s_2,((0,0,0)),q},\rho,x,\mathfrak{X}), \\
\mathscr{F}_3 &= \Lambda(s_1,l,w,\succ,\Psi_{s_1,((0,1,0)),q},\rho,x,\mathfrak{X}).
\end{aligned}$$

These $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$ are $\succ$-merged-log-concave. Then, Theorem 10.3 gives the $\succ$-merged-log-concave

$$\mathscr{F}_4 = \mathscr{F}_1 \circ \mathscr{F}_2 = \Lambda(s_2,l,w,\succ,\chi_{s_2,q},\rho,x,\mathfrak{X}).$$

Moreover, let $s_3 = s_1 + s_2 = (1,4)$. Then, Claims 3a and 4b of Lemma 15.2 and Theorem 15.25 yields the $\succ$-merged-log-concave

$$\mathscr{F}_5 = \mathscr{F}_3 * \mathscr{F}_4 = \Lambda(s_3,l,w,\succ,f_{5,s_3},\rho,x,\mathfrak{X})$$

with the following q-polynomials:

$$f_{5,s_3,(1)} = 1; \tag{16.2.1}$$
$$f_{5,s_3,(2)} = q^3+2q^2+2q+1; \tag{16.2.2}$$
$$f_{5,s_3,(3)} = 2q^6+5q^5+8q^4+7q^3+4q^2+q; \tag{16.2.3}$$
$$f_{5,s_3,(4)} = q^{11}+3q^{10}+7q^9+11q^8+14q^7+14q^6+11q^5+7q^4+3q^3+q^2. \tag{16.2.4}$$

16.2.1 On critical points and phase transitions

For $\theta = (2,3)$ and $P = r_{(1,\infty),\theta}(\pi(1))$, we have the ideal merged pair $\zeta = (P, \mathscr{F}_5)$. First, ζ has no front critical points, because $0<q<1$ does not solve

$$\mathscr{F}_{5,(1)}(q) = \mathscr{F}_{5,(2)}(q)$$

by equations (16.2.1) and (16.2.2). Second, ζ has the unique rear critical point $\mathrm{R_C}(\zeta) = 0.618034\ldots$ that solves

$$\mathscr{F}_{5,(3)}(q) = \mathscr{F}_{5,(4)}(q)$$

by equations (16.2.3) and (16.2.4). Then, we have the rear phase transition by Lemma 13.25. Third, there are no asymptotic critical points of ζ for the finite gate θ.

16.2.2 Polynomials with positive integer coefficients of an ideal merged pair

Let $P = ((s_1,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$ with $m_1=n_1=(1)$, $m_4=n_4=(4)$, and $k_1=k_4=k_2$. Then, we have the following merged determinants of ζ:

$$\begin{aligned}
\Delta(\mathscr{F}_5)(s_3,l,w,m_1,n_1,k_1,\rho,x,\mathfrak{X}) &= q^2+2q+1;\\
\Delta(\mathscr{F}_5)(s_3,l,w,m_2,n_2,k_2,\rho,x,\mathfrak{X}) &= q^{10}+6q^9+17q^8+31q^7+41q^6+42q^5+35q^4+24q^3+13q^2+5q+1;\\
\Delta(\mathscr{F}_5)(s_3,l,w,m_3,n_3,k_3,\rho,x,\mathfrak{X}) &= 3q^{18}+21q^{17}+81q^{16}+219q^{15}+456q^{14}+768q^{13}+1074q^{12}+1266q^{11}\\
&\quad+1266q^{10}+1074q^9+768q^8+456q^7+219q^6+81q^5+21q^4+3q^3;\\
\Delta(\mathscr{F}_5)(s_3,l,w,m_4,n_4,k_4,\rho,x,\mathfrak{X}) &= q^{30}+8q^{29}+38q^{28}+132q^{27}+369q^{26}+870q^{25}+1782q^{24}\\
&\quad+3232q^{23}+5260q^{22}+7754q^{21}+10423q^{20}+12836q^{19}+14527q^{18}\\
&\quad+15136q^{17}+14527q^{16}+12836q^{15}+10423q^{14}+7754q^{13}+5260q^{12}\\
&\quad+3232q^{11}+1782q^{10}+870q^9+369q^8+132q^7+38q^6+8q^5+q^4.
\end{aligned}$$

In particular, $\Delta(\mathscr{F}_5)(s_3,l,w,m_i,n_i,k_i,\rho,x,\mathfrak{X})$ is a log-concave q-polynomial for each $i\in[\![e(\theta)]\!]$ and palindromic for each $i\in[\![e(\theta)]\!]$ except $i\neq 2$.

17 Primal monomial parcels

Definition 17.1. *Suppose a parcel $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\rho,x,\mathfrak{X})$. We call $\mathscr{F}$ primal if $s=(0,\infty)$, $l=1$, $w=(1)$, $\mathscr{F}_{(0)}=1$, and x is fully $\succ$-admissible.*

The primal parcels are important for the theory of the merged-log-concavity. First, they construct parcels of arbitrary gates, widths, and weights by cut and shift operators and separable and Hadamard products. Second, they allow arbitrary base shifts by canonical mediators. Third, they consist of primal monomial parcels $\Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\rho,x,\mathfrak{X})$ such that $\gamma_{1,3}=0$.

17.1 Primal monomial parcels and q-dilogarithms

The generating functions of primal monomial parcels are quantum dilogarithms in [FadKas, FadVol, Kir, KonSoi, Rom, Schu, Zag]. This is because the generating functions satisfy pentagon identities in the following theorem, which is due to [FadKas, FadVol, Schu] (see also [Kir, Zag]). For completeness of this manuscript, we provide a full proof.

Theorem 17.2 ([FadKas, FadVol, Schu])**.** *Let $Q = \mathbb{Q}(\mathfrak{X})$ and $q \in Q$. For $u_0 = 1$ and $u_1 \in Q^\times$, consider a formal power series $k_q(t) = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} u_\lambda t^\lambda \in Q[[t]]$. Let $\tilde{Q} = Q\langle z_1, z_2\rangle / \langle z_1 z_2 = q z_2 z_1\rangle$. Also, let $p = 1 - q$ and $\mu = u_1 p$.*

1. *The following statements are equivalent.*
   (a) *We have the $(-\mu)$-pentagon identity $k_q(z_1)k_q(z_2) = k_q(z_2)k_q(-\mu z_2 z_1)k_q(z_1)$ in $\tilde{Q}$.*
   (b) $k_q(t) = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{\mu^\lambda}{(\lambda)_q} \cdot t^\lambda$.
2. *The following statements are equivalent.*
   (a) *We have the μ-pentagon identity $k_q(z_2)k_q(z_1) = k_q(z_1)k_q(\mu z_2 z_1)k_q(z_2)$ in $\tilde{Q}$.*
   (b) $k_q(t) = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{q^{\frac{\lambda(\lambda-1)}{2}} \cdot \mu^\lambda}{(\lambda)_q} \cdot t^\lambda$.

Proof. Proof of Claim 1. First, assume the $(-u_1)$-pentagon identity. By $z_1 z_2 = q z_2 z_1$, both sides of the $(-\mu)$-pentagon identity read

$$k_q(z_1)k_q(z_2) = \sum_{v_1, v_2 \in \mathbb{Z}_{\geq 0}} u_{v_1} u_{v_2} q^{v_1 v_2} z_2^{v_2} z_1^{v_1}, \tag{17.1.1}$$

$$\begin{aligned} k_q(z_2)k_q(-\mu z_2 z_1)k_q(z_1) &= \sum_{v'_1, v'_2, v'_3 \in \mathbb{Z}_{\geq 0}} u_{v'_1} u_{v'_2} u_{v'_3} z_2^{v'_1} (-\mu z_2 z_1)^{v'_2} z_1^{v'_3} \\ &= \sum_{v'_1, v'_2, v'_3 \in \mathbb{Z}_{\geq 0}} u_{v'_1} u_{v'_2} u_{v'_3} (-\mu)^{v'_2} q^{\frac{v'_2(v'_2-1)}{2}} z_2^{v'_1+v'_2} z_1^{v'_2+v'_3}. \end{aligned} \tag{17.1.2}$$

To obtain the term $z_2^{v_2} z_1^1$ in equation (17.1.2), we only have the following two cases: first, $v'_1 = v_2$, $v'_2 = 0$, and $v'_3 = 1$; second, $v'_1 = v_2 - 1$, $v'_2 = 1$, and $v'_3 = 0$. Both cases satisfy $q^{\frac{v'_2(v'_2-1)}{2}} = 1$. Hence, we have $u_1 u_{v_2} q^{v_2} = u_{v_2} u_1 - u_{v_2-1} u_1 \mu$, comparing coefficients of $z_2^{v_2} z_1^1$ in equations (17.1.1) and (17.1.2). Then, since $\mu u_{v_2-1} = u_{v_2}(1 - q^{v_2})$ by $u_1 \in Q^\times$, $u_{v_2-1} \frac{\mu}{1-q^{v_2}} = u_{v_2}$ implies Statement 1b.

Second, suppose Statement 1b. Then, both sides of the $(-\mu)$-pentagon identity read

$$k_q(z_1)k_q(z_2) = \sum_{v_1, v_2 \in \mathbb{Z}_{\geq 0}} \frac{\mu^{v_1+v_2}}{(v_1)_q (v_2)_q} q^{v_1 v_2} z_2^{v_2} z_1^{v_1}, \tag{17.1.3}$$

$$k_q(z_2)k_q(-\mu z_2 z_1)k_q(z_1) = \sum_{v'_1, v'_2, v'_3 \in \mathbb{Z}_{\geq 0}} (-1)^{v'_2} q^{\frac{v'_2(v'_2-1)}{2}} \frac{\mu^{v'_1+2v'_2+v'_3}}{(v'_1)_q (v'_2)_q (v'_3)_q} z_2^{v'_1+v'_2} z_1^{v'_2+v'_3}. \tag{17.1.4}$$

Now, $v'_1 + v'_2 = v_2$, $v'_2 + v'_3 = v_1$, and $v'_2 = \lambda$ imply $v'_1 = v_2 - \lambda \geq 0$ and $v'_3 = v_1 - \lambda \geq 0$. Thus, comparing coefficients of $z_2^{v_2} z_1^{v_1}$ in equations (17.1.3) and (17.1.4), the $(-\mu)$-pentagon identity follows from

$$\frac{q^{v_1 v_2}}{(v_1)_q (v_2)_q} = \sum_{\lambda \in [\![0, \min(v_1, v_2)]\!]} \frac{(-1)^\lambda q^{\frac{\lambda(\lambda-1)}{2}}}{(v_1 - \lambda)_q (\lambda)_q (v_2 - \lambda)_q}.$$

Multiplied by $(v_1)_q(v_2)_q$ on both sides, this is equivalent to

$$q^{v_1 v_2} = \sum_{\lambda \in [\![0,\min(v_1,v_2)]\!]} \begin{bmatrix} v_2 \\ \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ \lambda \end{bmatrix} (\lambda)_q (-1)^\lambda q^{\frac{\lambda(\lambda-1)}{2}}. \tag{17.1.5}$$

Let us prove equation (17.1.5). Assume $v_1 \geq v_2$ without loss of the generality. When $v_2 = 0$, equation (17.1.5) clearly holds. Furthermore, the induction on v_2 gives

$$\sum_{\lambda \in [\![0,v_2-1]\!]} \begin{bmatrix} v_2 - 1 \\ \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ \lambda \end{bmatrix} (\lambda)_q (-1)^\lambda q^{\frac{\lambda(\lambda-1)}{2}} = q^{v_1(v_2-1)},$$
$$\sum_{\lambda \in [\![0,v_2-1]\!]} \begin{bmatrix} v_2 - 1 \\ \lambda \end{bmatrix} \begin{bmatrix} v_1 - 1 \\ \lambda \end{bmatrix} (\lambda)_q (-1)^\lambda q^{\frac{\lambda(\lambda-1)}{2}} = q^{(v_1-1)(v_2-1)}.$$

Thus, we obtain equation (17.1.5), as the q-Pascal identity gives

$$\begin{aligned}
\sum_{\lambda \in [\![0,v_2]\!]} \begin{bmatrix} v_2 \\ \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ \lambda \end{bmatrix} (\lambda)_q (-1)^\lambda q^{\frac{\lambda(\lambda-1)}{2}} &= \sum_{\lambda \in [\![0,v_2]\!]} \begin{bmatrix} v_2 - 1 \\ \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ \lambda \end{bmatrix} (\lambda)_q (-1)^\lambda q^{\frac{\lambda(\lambda-1)}{2}} \\
&\quad + \sum_{\lambda \in [\![v_2]\!]} q^{v_2-\lambda} \begin{bmatrix} v_2 - 1 \\ \lambda - 1 \end{bmatrix} \begin{bmatrix} v_1 \\ \lambda \end{bmatrix} (\lambda)_q (-1)^\lambda q^{\frac{\lambda(\lambda-1)}{2}} \\
&= \sum_{\lambda \in [\![0,v_2-1]\!]} \begin{bmatrix} v_2 - 1 \\ \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ \lambda \end{bmatrix} (\lambda)_q (-1)^\lambda q^{\frac{\lambda(\lambda-1)}{2}} \\
&\quad + \sum_{\lambda \in [\![v_2]\!]} q^{(v_2-1)-(\lambda-1)} \frac{(1-q^{v_1})}{(1-q^\lambda)} \begin{bmatrix} v_2 - 1 \\ \lambda - 1 \end{bmatrix} \begin{bmatrix} v_1 - 1 \\ \lambda - 1 \end{bmatrix} (\lambda)_q (-1)^\lambda q^{\frac{\lambda(\lambda-1)}{2}} \\
&= q^{v_1(v_2-1)} \\
&\quad + (1-q^{v_1}) q^{v_2-1} \sum_{\lambda \in [\![v_2]\!]} \begin{bmatrix} v_2 - 1 \\ \lambda - 1 \end{bmatrix} \begin{bmatrix} v_1 - 1 \\ \lambda - 1 \end{bmatrix} (\lambda-1)_q (-1)^\lambda q^{\frac{(\lambda-1)(\lambda-2)}{2}} \\
&= q^{v_1(v_2-1)} \\
&\quad + (q^{v_1}-1) q^{v_2-1} \sum_{\lambda \in [\![0,v_2-1]\!]} \begin{bmatrix} v_2 - 1 \\ \lambda \end{bmatrix} \begin{bmatrix} v_1 - 1 \\ \lambda \end{bmatrix} (\lambda)_q (-1)^\lambda q^{\frac{\lambda(\lambda-1)}{2}} \\
&= q^{v_1(v_2-1)} + (q^{v_1}-1) q^{v_2-1} q^{(v_1-1)(v_2-1)} \\
&= q^{v_1 v_2}.
\end{aligned}$$

Proof of Claim 2. First, suppose Statement 2a. By $u_0 = 1$, $k_q(t) \in Q[[t]]^\times$. Let $\omega_q(t) = k_q(t)^{-1} = 1 - u_1 t + \dots$. Then, inverting both sides of the μ-pentagon identity, we have

$$\omega_q(z_1)\omega_q(z_2) = \omega_q(z_2)\omega_q(-(-\mu)pz_2z_1)\omega_q(z_1). \tag{17.1.6}$$

Hence, Claim 1 implies $\omega_q(t) = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{(-\mu)^\lambda}{(\lambda)_q} t^\lambda$. By the Euler binomial identities, Statement 2b follows from

$$k_q(t) = \omega_q(t)^{-1} = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{(-1)^\lambda q^{\frac{\lambda(\lambda-1)}{2}} (-\mu)^\lambda}{(\lambda)_q} t^\lambda = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{q^{\frac{\lambda(\lambda-1)}{2}} \mu^\lambda}{(\lambda)_q} t^\lambda.$$

Second, suppose Statement 2b. Then, we go backward the discussion above, as Claim 1 is an equivalence. This gives equation (17.1.6) for $\omega_q(t) = k_q(t)^{-1}$. Thus, $k_q(t)$ satisfies the μ-pentagon identity. $\square$

In Theorem 17.2, $k_q(t)$ that satisfies one of the pentagon identities is a quantum dilogarithm. In particular, we adopt the following notation by q-Pochhammer symbols.

Definition 17.3. *Let $\kappa \in \mathbb{Q}$ and $q \in \mathbb{Q}(\mathfrak{X})$. We call $k_q(t) \in \mathbb{Q}(\mathfrak{X})[[t]]$ a Pochhammer quantum dilogarithm, if*

$$k_q(t) = \begin{cases} \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{q^{\kappa\lambda}}{(\lambda)_q} \cdot t^\lambda \text{ or} \\ \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{q^{\frac{\lambda(\lambda-1)}{2}+\kappa\lambda}}{(\lambda)_q} \cdot t^\lambda. \end{cases}$$

These Pochhammer quantum dilogarithms have been studied intensively. For instance, $(x;q)_\infty$ in [FadKas] is $k_q(t) = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{t^{\frac{\lambda(\lambda-1)}{2}}}{(\lambda)_q}$ for $x = -t$. Also, $1 + \sum_{\lambda \in \mathbb{Z}_{\geq 1}} \frac{q^{\frac{\lambda^2}{2}}}{(q^\lambda - q^{\lambda-1})\ldots(q^\lambda - q)(q^\lambda - 1)} x^\lambda$ in [KonSoi, Section 6.4] is $k_q(t) = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{q^{\frac{\lambda}{2}}}{(\lambda)_q} t^\lambda$. Furthermore, we state the following.

Proposition 17.4. *Suppose a primal monomial parcel $\mathscr{F} = \Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\rho,x,\mathfrak{X})$. Then, $\mathscr{Z}_{\gamma,q}(t)$ is a Pochhammer quantum dilogarithm.*

Proof. The monomial conditions of $\mu = (l,w,\gamma)$ imply $\gamma_{1,1} = 0$ or $\gamma_{1,1} = \frac{1}{2}$. If $\gamma_{1,1} = 0$, then $\mathscr{Z}_{\gamma,q}(t) = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{q^{\gamma_{1,2}\lambda}}{(\lambda)_q} \cdot t^\lambda \in \mathbb{Q}(\mathfrak{X})[[t]]$. Also, if $\gamma_{1,1} = \frac{1}{2}$, then $\mathscr{Z}_{\gamma,q}(t) = \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{q^{\frac{\lambda^2}{2}+\gamma_{1,2}\lambda}}{(\lambda)_q} \cdot t^\lambda \in \mathbb{Q}(\mathfrak{X})[[t]]$. □

17.2 Phase transitions and the golden ratio

We now study the almost strictly unimodal sequences and phase transitions. This gives the golden ratio as a critical point. We introduce the following notion for merged pairs.

Definition 17.5. *Suppose the merged pair $\zeta_P = (P,\mathscr{F})$ of a parcel $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ and fitting path $P = ((s,l,m_i,n_i,k_i))_{i \in [\![\theta]\!]}$.*

1. *We call ζ_P vanishing if $\lim_{i\to\infty} u(\zeta,r)_i = 0$ for each $r \in \mathrm{O}_{\mathfrak{X}}$.*
2. *We call ζ_P probabilistic if $\sum_{i \in [\![e(\theta)]\!]} u(\zeta,r)_i < \infty$ for each $r \in \mathrm{O}_{\mathfrak{X}}$.*

If ζ_P is vanishing (resp. probabilistic) for each fitting path P, then we call $\mathscr{F}$ vanishing (resp. probabilistic).

We have the following equivalence.

Proposition 17.6. *Suppose a primal monomial parcel $\mathscr{F} = \Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\rho,x,\mathfrak{X})$. Let $r \in \mathrm{O}_{\mathfrak{X}}$. Then, the following statements are equivalent.*

1. *$\mathscr{F}$ is vanishing.*
2. *There exists some $\lambda \in \mathbb{Z}_{\geq 1}$ such that $t_\gamma((\mu)) > 0$ for each $\mu \in \mathbb{Z}_{\geq \lambda}$.*
3. *There exist real numbers $N_\gamma(r) \geq 1$ and $0 < S_\gamma(r) < 1$ such that $\mathscr{F}_{(\mu)}(r) \leq N_\gamma(r) S_\gamma(r)^\mu$ for each $\mu \in \mathbb{Z}_{\geq 0}$.*
4. *$\mathscr{F}$ is probabilistic.*

Proof. We prove Statement 2 from Statement 1. Since $w=(1)$, the monomial conditions of $\kappa=(l,w,\gamma)$ imply $\gamma_{1,1}=\frac{1}{2}$ or 0. Suppose $\gamma_{1,1}=0$. If $\gamma_{1,2}\leq 0$, then

$$\mathscr{F}_{(0)}(r)=1<\mathscr{F}_{(1)}(r)=\frac{q(r)^{\gamma_{1,2}}}{1-q(r)}<\mathscr{F}_{(2)}(r)=\frac{q(r)^{2\gamma_{1,2}}}{(1-q(r))(1-q(r)^2)}<\dots,$$

since $0<q(r)<1$ in Claim 1 of Lemma 5.22. However, this is against Statement 1, which makes $\Omega(\mathscr{F})$ vanishing. Hence, Statement 2 follows from $\gamma_{1,2}>0$. If $\gamma_{1,1}=\frac{1}{2}$, then Statement 2 holds for any $\gamma_{1,2}\in\mathbb{Q}$.

We prove Statement 3 from Statement 2. Since $\gamma_{1,1}=0$ and $\gamma_{1,2}>0$, or $\gamma_{1,1}=\frac{1}{2}$, we have $U_\gamma\in\mathbb{Q}_{>0}$ and $V_\gamma(r)\in\mathbb{Z}_{\geq 0}$ such that each integer $\mu\geq V_\gamma(r)+1$ satisfies

$$\frac{q(r)^{U_\gamma}}{1-q(r)^{V_\gamma(r)+1}}<1, \tag{17.2.1}$$

$$t_\gamma((\mu))\geq U_\gamma\mu. \tag{17.2.2}$$

Let $\upsilon(r)=\frac{1}{1-q(r)}$, $S_\gamma(r)=\frac{q(r)^{U_\gamma}}{1-q(r)^{V_\gamma(r)+1}}$, and $\chi_\gamma(r)=\frac{\upsilon(r)}{S_\gamma(r)}>1$. In particular, $\mu\in\mathbb{Z}_{\geq 1}$ gives

$$\upsilon(r)>\frac{q(r)^{U_\gamma}}{1-q(r)^{\mu}}, \tag{17.2.3}$$

$$S_\gamma(r)\geq\frac{q(r)^{U_\gamma}}{1-q(r)^{V_\gamma(r)+\mu}}. \tag{17.2.4}$$

Also, let $\omega_\gamma(r)=\max(q(r)^{t_\gamma((0))}=1,\cdots,q(r)^{t_\gamma((V_\gamma(r)))})$ and $N_\gamma(r)=\left(\omega_\gamma(r)\chi_\gamma(r)\right)^{V_\gamma(r)}\geq 1$.

First, assume $\mu\geq V_\gamma(r)+1$. Then, inequality (17.2.2) gives

$$\mathscr{F}_{(\mu)}(r)=\frac{q(r)^{t_\gamma((\mu))}}{(\mu)_q|_{q=q(r)}}\leq\prod_{i\in[\![\mu]\!]}\frac{q(r)^{U_\gamma}}{1-q(r)^i}.$$

Hence,

$$\mathscr{F}_{(\mu)}(r)\leq\upsilon(r)^{V_\gamma(r)}S_\gamma(r)^{\mu-V_\gamma(r)}=\chi_\gamma(r)^{V_\gamma(r)}S_\gamma(r)^{\mu}\leq N_\gamma(r)S_\gamma(r)^{\mu}$$

by inequalities (17.2.3) and (17.2.4).

Second, suppose $0\leq\mu\leq V_\gamma(r)$. Then,

$$\mathscr{F}_{(\mu)}(r)\leq\left(\frac{\omega_\gamma(r)}{1-q(r)}\right)^{\mu}\leq\left(\omega_\gamma(r)\upsilon(r)\right)^{V_\gamma(r)}.$$

Thus, inequality (17.2.1) gives

$$\mathscr{F}_{(\mu)}(r)\leq(\omega_\gamma(r)\upsilon(r))^{V_\gamma(r)}\cdot\frac{S_\gamma(r)^{\mu}}{S_\gamma(r)^{V_\gamma(r)}}=N_\gamma(r)S_\gamma(r)^{\mu}.$$

Statement 4 follows from Statement 3, since $\sum_{m\in[\![s]\!]^l}\mathscr{F}_m(s)\leq N_\gamma(r)\sum_{i\in\mathbb{Z}_{\geq 0}}S_\gamma(r)^i<\infty$. Also, Statement 4 implies Statement 1 by $\mathscr{F}_{(\mu)}(r)>0$. □

We discuss phase transitions first by asymptotic critical points.

Lemma 17.7. *Consider the merged pair $\zeta = (P, \mathscr{F})$ of an infinite-length P and a primal monomial parcel $\mathscr{F} = \Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\rho,x,\mathfrak{X})$. When $r \in \mathrm{O}_{\mathfrak{X}}$, we have the following.*

1. *If $\gamma_{1,1} = \gamma_{1,2} = 0$, then $u(\zeta,r)$ is an asymptotic hill.*
2. *If not, then ζ has no asymptotic critical points.*

Proof. Let $P = ((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$ of an infinite θ. Let $\theta_1 = 1$ for simplicity. Then, Lemma 13.33 provides $\kappa \in \mathbb{Z}_{\geq 1}$ and $h \in \mathbb{Z}$ such that $m_i = n_i = (i\kappa + h)$ for $i \in [\![\theta]\!]$.

Proof of Claim 1. Claim 1 holds by Proposition 13.7, since Claim 1 of Lemma 5.22 gives

$$\lim_{j\to\infty} \frac{\mathscr{F}_{m_{j+1}}(r)}{\mathscr{F}_{m_j}(r)} = \lim_{j\to\infty} \frac{(j\kappa+h)_q|_{q=q(r)}}{((j+1)\kappa+h)_q|_{q=q(r)}} = \lim_{j\to\infty} \prod_{i\in[\![\lambda]\!]} \frac{1}{1-q(r)^{j\kappa+h+i}} = 1.$$

Proof of Claim 2. First, suppose $\lambda \in \mathbb{Z}_{\geq 1}$ such that each $\mu \geq \lambda$ satisfies

$$t_\gamma((\mu)) > 0. \tag{17.2.5}$$

Then, Lemma 12.27 and Proposition 17.6 yield $\lim_{j\to\infty} u(\zeta,r)_j = 0$. Thus, Claim 2 follows, since $u(\zeta,r) > 0$ is a hill or strictly decreasing sequence by Claim 2 of Theorem 12.18.

Second, assume that inequality (17.2.5) does not hold. This gives $\gamma_{1,1} = 0$ and $\gamma_{1,2} \leq 0$, since $\gamma_{1,1} = \frac{1}{2}$ or 0 by the monomial conditions. Thus, the assumption of Claim 2 implies $\gamma_{1,1} = 0$ and $\gamma_{1,2} < 0$. Then, $q(r)^{\gamma_{1,2}} > 1$ by Claim 1 of Lemma 5.22. Claim 2 now holds, since

$$\lim_{j\to\infty} \frac{\mathscr{F}_{m_{j+1}}}{\mathscr{F}_{m_j}} = \lim_{j\to\infty} \frac{q(r)^{((j+1)\kappa+h)\cdot\gamma_{1,2}}}{((j+1)\kappa+h)_q|_{q=q(r)}} \cdot \frac{(j\kappa+h)_q|_{q=q(r)}}{q(r)^{(j\kappa+h)\cdot\gamma_{1,2}}} = \lim_{j\to\infty} \frac{q(r)^{\kappa\gamma_{1,2}}}{\prod_{i\in[\![\kappa]\!]}(1-q(r)^{j\kappa+h+i})} > 1.$$

□

By the following notation, we obtain front and rear phase transitions.

Definition 17.8. *Let $\theta = (1,\infty)$ and $\chi \geq 1$ be a gate. Suppose a merged pair $\Omega_\lambda(\mathscr{F})$. Then, let*

$$\Omega^\chi_\lambda(\mathscr{F}) = \left(r_{\theta,\chi}(\pi(\lambda)), \mathscr{F}\right),$$

which we call a restricted merged pair.

Lemma 17.9. *Let $\mathscr{F} = \Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\rho,x,\mathfrak{X})$ be a primal monomial parcel. Suppose some $\lambda \in \mathbb{Z}_{\geq 1}$ such that each $\mu \in \mathbb{Z}_{\geq \lambda}$ satisfies*

$$t_\gamma((\mu)) > 0. \tag{17.2.6}$$

Then, we have the following.

1. *The merged pair $\zeta_\lambda = \Omega_\lambda(\mathscr{F})$ has a front phase transition.*
2. *The merged pair $\zeta_{\lambda,\chi} = \Omega^\chi_\lambda(\mathscr{F})$ has a rear phase transition for each finite gate χ.*

Proof. For $\theta = (1,\infty)$, suppose the fitting path $\pi(\lambda) = ((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$ with $m_0 = n_0 = (0)$. Let us use Lemma 13.25.

Proof of Claim 1. There is $d \in \mathbb{Z}_{\geq 1}$ such that $T = q^{\frac{1}{d}} \in \mathfrak{X}$ for the fully admissible q. Then, we put

$$\begin{aligned}\mathrm{Bot}_1(T) &= \mathrm{Top}_1(T) = 1,\\ \mathrm{Bot}_2(T) &= (\lambda)_q,\\ \mathrm{Top}_2(T) &= q^{t_\gamma((\lambda))}.\end{aligned}$$

In particular, $\mathrm{Frac}(\mathrm{Top})(T) = q^{t_\gamma((\lambda))}$ and $\mathrm{Frac}(\mathrm{Bot})(T) = (\lambda)_q$. Then, $\mathrm{Frac}(\mathrm{Top})(0) = 0$ by inequality (17.2.6) and $\mathrm{Frac}(\mathrm{Top})(1) = 1$. Also, we have $\mathrm{Frac}(\mathrm{Bot})(0) = 1$ and $\mathrm{Frac}(\mathrm{Bot})(1) = 0$. Thus, Lemma 13.25 gives Claim 1 by

$$\begin{aligned}\mathscr{F}_{(0)}(q) &= \frac{\mathrm{Top}_1(T)}{\mathrm{Bot}_1(T)},\\ \mathscr{F}_{(\lambda)}(q) &= \frac{\mathrm{Top}_2(T)}{\mathrm{Bot}_2(T)}.\end{aligned}$$

Proof of Claim 2. Let $\kappa \in \mathbb{Z}_{\geq 1}$ and $\chi = (1,\kappa)$, Also, we put

$$\begin{aligned}\mathrm{Bot}_1(T) &= (\kappa\lambda)_q,\\ \mathrm{Bot}_2(T) &= ((\kappa+1)\lambda)_q,\\ \mathrm{Top}_1(T) &= q^{t_\gamma((\kappa\lambda))},\\ \mathrm{Top}_2(T) &= q^{t_\gamma(((\kappa+1)\lambda))}.\end{aligned}$$

Then, we have $\mathrm{Frac}(\mathrm{Bot})(0) = 1$, $\mathrm{Frac}(\mathrm{Bot})(1) = 0$, and $\mathrm{Frac}(\mathrm{Top})(1) = 1$. Also, $\mathrm{Frac}(\mathrm{Top})(0) = 0$, because inequality (17.2.6) and $\gamma_{1,1} \geq 0$ imply

$$\begin{aligned}t_\gamma(((\kappa+1)\lambda)) - t_\gamma((\kappa\lambda)) &= (((\kappa+1)\lambda)^2 - (\kappa\lambda)^2)\gamma_{1,1} + ((\kappa+1)\lambda - \kappa\lambda)\gamma_{1,2}\\ &= (2\kappa\lambda + \lambda^2)\gamma_{1,1} + \lambda\gamma_{1,2}\\ &= 2\kappa\gamma_{1,1} + t_\gamma((\lambda))\\ &> 0.\end{aligned}$$

□

We have the following positive values of t_γ by semi-phase transitions.

Lemma 17.10. *Consider a primal monomial parcel $\mathscr{F} = \Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\rho,x,\mathfrak{X})$. Let $\lambda \in \mathbb{Z}_{\geq 1}$.*

1. *If $\Omega_\lambda(\mathscr{F})$ has a front semi-phase transition, then $t_\gamma((\mu)) > 0$ for each $\mu \in \mathbb{Z}_{\geq \lambda}$.*
2. *For a finite gate χ, if $\Omega^\chi_\lambda(\mathscr{F})$ has a rear semi-phase transition, then there is $\lambda' \in \mathbb{Z}_{\geq 1}$ such that $t_\gamma((\mu)) > 0$ for each $\mu \in \mathbb{Z}_{\geq \lambda'}$.*

Proof. For $\theta = (1,\infty)$, let $\pi(\lambda) = ((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$ with $m_0 = n_0 = (0)$. Also, let $\kappa = (l,w,\gamma)$.

Proof of Claim 1. We have $0 < (\lambda)_q|_{q=q(r)} < 1$ by Claim 1 of Lemma 5.22. Also, the front semi-phase transition gives $r \in \mathrm{O}_{\mathfrak{X}}$ such that

$$\mathscr{F}_{m_0}(r) = 1 = \frac{q(r)^{t_\gamma((\lambda))}}{(\lambda)_q|_{q=q(r)}} = \mathscr{F}_{m_1}(r).$$

Hence, $0 < q(r)^{t_\gamma((\lambda))} < 1$, which implies

$$t_\gamma((\lambda)) = \gamma_{1,1}\lambda^2 + \gamma_{1,2}\lambda > 0.$$

Claim 1 now holds, since $\gamma_{1,1} \geq 0$ by the monomial conditions of κ.

Proof of Claim 2. The rear semi-phase transition demands $r \in \mathrm{O}_{\mathfrak{X}}$ such that

$$\mathscr{F}_{m_{\chi_2}}(r) = \frac{q(r)^{t_\gamma((\chi_2\lambda))}}{(\chi_2\lambda)_q|_{q=q(r)}} = \frac{q(r)^{t_\gamma(((\chi_2+1)\lambda))}}{((\chi_2+1)\lambda)_q|_{q=q(r)}} = \mathscr{F}_{m_{\chi_2+1}}(r).$$

Then, $0 < q(r)^{t_\gamma(((\chi_2+1)\lambda)) - t_\gamma((\chi_2\lambda))} < 1$ by $0 < \frac{((\chi_2+1)\lambda)_q|_{q=q(r)}}{(\chi_2\lambda)_q|_{q=q(r)}} < 1$. In particular,

$$t_\gamma(((\chi_2+1)\lambda)) - t_\gamma((\chi_2\lambda)) = (2\chi_2\lambda + \lambda^2)\gamma_{1,1} + \lambda\gamma_{1,2} > 0.$$

Thus, the monomial conditions of κ imply $\gamma_{1,1} = \frac{1}{2}$, or $\gamma_{1,1} = 0$ and $\gamma_{1,2} > 0$. □

Then, we obtain the following on the phase transitions of primal monomial parcels. This extends Proposition 13.38 by vanishing and probabilistic parcels.

Theorem 17.11. *Suppose a primal monomial parcel $\mathscr{F} = \Lambda(s, l, w, \succ, \Psi_{s,\gamma,q}, \rho, x, \mathfrak{X})$. Then, we have the following.*

1. *Each merged pair $(P, \mathscr{F})$ has no asymptotic semi-phase transitions.*

2. *The following statements are equivalent.*

   (a) *$\mathscr{F}$ is vanishing.*
   (b) *$\mathscr{F}$ is probabilistic.*
   (c) *There exists some $\lambda \in \mathbb{Z}_{\geq 1}$ such that $t_\gamma((\mu)) > 0$ for each $\mu \in \mathbb{Z}_{\geq \lambda}$.*
   (d) *For some $\lambda \in \mathbb{Z}_{\geq 1}$, $\Omega_\lambda(\mathscr{F})$ has a front phase transition.*
   (e) *For some $\lambda \in \mathbb{Z}_{\geq 1}$, $\Omega_\lambda(\mathscr{F})$ has a front semi-phase transition.*
   (f) *For some $\lambda \in \mathbb{Z}_{\geq 1}$ and finite gate χ, $\Omega^\chi_\lambda(\mathscr{F})$ has a rear phase transition.*
   (g) *For some $\lambda \in \mathbb{Z}_{\geq 1}$ and finite gate χ, $\Omega^\chi_\lambda(\mathscr{F})$ has a rear semi-phase transition.*

3. *If one of Statements 2c – 2e holds by some $\lambda \in \mathbb{Z}_{\geq 1}$, then each of Statements 2c – 2g holds by the same $\lambda \in \mathbb{Z}_{\geq 1}$.*

Proof. Proof of Claim 1. Claim 1 holds by Lemma 17.7.

Proof of Claim 2. We obtain Claim 2 by the following.

- Statements 2a, 2b, and 2c are equivalent by Proposition 17.6.
- Statements 2c, 2d, and 2e are equivalent by Claim 1 of Lemma 17.9 and Claim 1 of Lemma 17.10.
- Statements 2c, 2f, and 2g are equivalent by Claim 2 of Lemma 17.9 and Claim 2 of Lemma 17.10.

Proof of Claim 3. Claim 1 of Lemma 17.9 and Claim 1 of Lemma 17.10 imply that Statements 2c, 2d, and 2e hold by the same λ, which give Statements 2f and 2g by Claim 2 of Lemma 17.9. □

Furthermore, Theorem 17.11 gives the following polynomials with positive integer coefficients by the finest fitting path $\pi(1)$.

Proposition 17.12. *Let $\mathscr{F} = \Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\rho,x,\mathfrak{X})$ be a primal monomial parcel. Then, Statements 1 and 2 below are equivalent.*

1. *The merged pair $\xi = \Omega(\mathscr{F})$ is ideal with a front phase transition.*
2. *We have some $\lambda \in \mathbb{Z}_{\geq 1}$ such that*

$$\gamma = \left(\left(0, \frac{\lambda}{2}, 0\right)\right), \tag{17.2.7}$$

or some $\lambda \in \mathbb{Z}_{\geq 0}$ such that

$$\gamma = \left(\left(\frac{1}{2}, \frac{\lambda}{2}, 0\right)\right). \tag{17.2.8}$$

Proof. For $\theta = (1,\infty)$, let $\pi(1) = ((s,l,m_i,n_i,k_i))_{i\in[\![\theta]\!]}$ with $a_i = \nu(k_i)$ and $b_i = \nu(m_i,n_i,k_i)$. Suppose $\phi(x) = (1-q) \in \mathbb{Q}(\mathfrak{X})^l$ and $i \in [\![\theta]\!]$. Then, Theorem 8.40 gives

$$\begin{aligned} q^{-2(\gamma_{1,1}i^2+\gamma_{1,2}i)}\Delta(\mathscr{F})(s,l,w,m_i,n_i,k_i,\phi,\rho,x,\mathfrak{X}) &= \tilde{d}(V)^{b_i}_{a_i}(s,l,w,\phi,\rho,t_{\gamma,\Delta}(m_i,n_i,k_i),x) \\ &>_q 0. \end{aligned} \tag{17.2.9}$$

We prove Statement 2 from Statement 1. Since $\Delta(\mathscr{F})(s,l,w,m_i,n_i,k_i,\phi,\rho,x,\mathfrak{X}) >_q 0$ for the ideal ξ, inequality (17.2.9) implies

$$2(\gamma_{1,1}i^2 + \gamma_{1,2}i) \in \mathbb{Z}. \tag{17.2.10}$$

First, assume $\gamma_{1,1} = 0$. Then, inclusion (17.2.10) gives $2\gamma_{1,2} \in \mathbb{Z}$ by $i = 1$. Thus, equation (17.2.7) holds by

$$t_\gamma((1)) = \gamma_{1,2} > 0$$

in Claim 3 of Theorem 17.11. Second, assume $\gamma_{1,1} = \frac{1}{2}$. Then, inclusion (17.2.10) implies $1 + 2\gamma_{1,2} \in \mathbb{Z}$ by $i = 1$. Hence, equation (17.2.8) follows from

$$t_\gamma((1)) = \frac{1}{2} + \gamma_{1,2} > 0$$

in Claim 3 of Theorem 17.11.

We prove Statement 1 from Statement 2. First, suppose equation (17.2.7). By inequality (17.2.9), ξ is ideal, since

$$2(\gamma_{1,1}i^2 + \gamma_{1,2}i) = \lambda i \in \mathbb{Z}_{\geq 1}.$$

Thus, Statement 1 holds by Claim 3 of Theorem 17.11, since any $\mu \in \mathbb{Z}_{\geq 1}$ satisfies

$$t_\gamma((\mu)) = \frac{\lambda}{2}\mu > 0.$$

Second, suppose equation (17.2.8). By inequality (17.2.9), ξ is ideal, since

$$2(\gamma_{1,1}i^2+\gamma_{1,2}i)=i^2+\lambda i\in\mathbb{Z}_{\geq 1}.$$

Claim 3 of Theorem 17.11 now gives Statement 1, since any $\mu\in\mathbb{Z}_{\geq 1}$ satisfies

$$t_\gamma((\mu))=\frac{\mu^2}{2}+\frac{\lambda\mu}{2}>0.$$

□

We introduce the following notation to compare merged pairs by bases and almost strictly unimodal sequences.

Definition 17.13. *For $i\in[\![2]\!]$, suppose parcels $\mathscr{F}_i=\Lambda(s,l,w,\succ_i,f_{i,s},\phi_i,\rho_i,x,\mathfrak{X}_i)$ so that each $x_j\in\mathbb{Q}(\mathfrak{X}_1)\cap\mathbb{Q}(\mathfrak{X}_2)$.*

1. *If $r_1\in A_{\mathfrak{X}_1}$ and $r_2\in A_{\mathfrak{X}_2}$ satisfy $x(r_1)=x(r_2)\in\mathbb{R}^l$, then we write $(\mathfrak{X}_1,r_1)\equiv^x(\mathfrak{X}_2,r_2)$.*
2. *Consider $(\theta,\mathfrak{X}_i)$-merged pairs $\zeta_i=(P,\mathscr{F}_i)$ for $i\in[\![2]\!]$. If $u(\zeta_1,r_1)\geq u(\zeta_2,r_2)$ whenever $(\mathfrak{X}_1,r_1)\equiv^x(\mathfrak{X}_2,r_2)$, then we say that ζ_1 covers ζ_2 on x and write $\zeta_1\geq^x\zeta_2$.*

Let us state the following reflexivity and transitivity of the covering relation $\geq^x$.

Lemma 17.14. *For $i\in[\![3]\!]$, consider the $(\theta,\mathfrak{X}_i)$-merged pairs $\zeta_i=(P,\mathscr{F}_i)$ of $\mathscr{F}_i=\Lambda(s,l,w,\succ_i,f_{i,s},\phi_i,\rho_i,x,\mathfrak{X}_i)$. Then, we have the following.*

1. *$\zeta_1\geq^x\zeta_1$.*
2. *$\zeta_1\geq^x\zeta_2$ and $\zeta_2\geq^x\zeta_3$ imply $\zeta_1\geq^x\zeta_3$.*

Proof. Proof of Claim 1. Claim 1 holds by $u(\zeta_1,r_1)\geq u(\zeta_1,r_1)$ for $r_1\in O_{\mathfrak{X}_1}$.

Proof of Claim 2. Claim 2 holds, since $(\mathfrak{X}_1,r_1)\equiv^x(\mathfrak{X}_2,r_2)\equiv^x(\mathfrak{X}_3,r_3)$ implies $u(\zeta_1,r_1)\geq u(\zeta_2,r_2)\geq u(\zeta_3,r_3)$. □

The antisymmetricity of $\geq^x$ does not hold in general. However, we state the following.

Lemma 17.15. *For $i\in[\![2]\!]$, consider the $(\theta,\mathfrak{X}_i)$-merged pairs $\zeta_i=\Omega(\mathscr{F}_i)$ of primal monomial parcels $\mathscr{F}_i=\Lambda(s,l,w,\succ_i,\Psi_{s,\gamma_i,q},\rho_i,x,\mathfrak{X}_i)$. If $\zeta_1\geq^x\zeta_2\geq^x\zeta_1$, then $\mathscr{F}_1=\mathscr{F}_2$.*

Proof. By the primal assumption, $q^{\frac{1}{d_1}}\in\mathfrak{X}_1$ and $q^{\frac{1}{d_2}}\in\mathfrak{X}_2$ for some $d_1,d_2\in\mathbb{Z}_{\geq 1}$. Then, the covering relations imply $\mathscr{F}_{1,(i)}(q^{\frac{1}{d_1}})=\mathscr{F}_{2,(i)}(q^{\frac{1}{d_2}})$ for each $i\in\mathbb{Z}_{\geq 0}$ and $0<q<1$. This implies

$$\mathscr{F}_{1,(i)}=\mathscr{F}_{2,(i)}\in\mathbb{Q}\left(q^{\frac{1}{d_1d_2}}\right)$$

so that $\mathscr{F}_{1,(i)}=\mathscr{F}_{2,(i)}$ have infinite solutions. □

We now identify the $q^{\frac{1}{2}}$-linear $\mathscr{L}$ in Definition 14.2 as the extremal parcel among primal monomial parcels by q-polynomials with positive integer coefficients and phase transitions. Also, we obtain the golden ratio of $\mathscr{L}$ as the critical point of $\Omega(\mathscr{L})$ (see Section 1.8).

Corollary 17.16. *Consider the ideal merged pair $\zeta = \Omega(\mathscr{L})$ of the the $q^{\frac{1}{2}}$-linear*

$$\mathscr{L} = \Lambda(s, l, w, \succ, \Psi_{s,\gamma,q}, \rho, x, \mathfrak{X}).$$

1. *Let $\mathscr{F} = \Lambda(s, l, w, \succ', \Psi_{s,\gamma',q}, \rho', x, \mathfrak{X}')$ be a primal monomial parcel and $\zeta' = \Omega(\mathscr{F})$. If ζ' has a front phase transition, then $\zeta \geq^x \zeta'$.*
2. *Consider all primal monomial parcels $\mathscr{F} = \Lambda(s, l, w, \succ', \Psi_{s,\gamma',q}, \rho', x, \mathfrak{X}')$ such that each $\Omega(\mathscr{F})$ has a front phase transition. Among them, $\mathscr{L}$ gives the unique maximum ideal merged pair ζ with respect to the covering relation $\geq^x$.*
3. *The single critical point of ζ is the golden ratio $\mathrm{F_C}(\zeta) = \frac{-1+\sqrt{5}}{2}$.*

Proof. Proof of Claim 1. By the existence of a front phase transition, Proposition 17.12 implies

$$\gamma' = \left(\left(0, \frac{\lambda}{2}, 0\right)\right) \text{ for some } \lambda \in \mathbb{Z}_{\geq 1}, \text{ or}$$
$$\gamma' = \left(\left(\frac{1}{2}, \frac{\lambda}{2}, 0\right)\right) \text{ for some } \lambda \in \mathbb{Z}_{\geq 0}.$$

Thus, each $i \in \mathbb{Z}_{\geq 0}$ satisfies

$$t_\gamma((i)) = \frac{i}{2} \leq t_{\gamma'}((i)).$$

Also, suppose $(\mathfrak{X}, r) \equiv^x (\mathfrak{X}', r')$. Then, $0 < q(r) = q(r') < 1$ in Claim 1 of Lemma 5.22. Hence,

$$u(\zeta, r)_i = \frac{q(r)^{t_\gamma((i))}}{(i)_q\big|_{q=q(r)}} \geq \frac{q(r')^{t_{\gamma'}((i))}}{(i)_q\big|_{q=q(r')}} = u(\zeta', r')_i.$$

Proof of Claim 2. Claim 2 follows from Lemma 17.15 and Claim 1.

Proof of Claim 3. A real number $0 < q^{\frac{1}{2}} < 1$ is a front critical point of ζ if and only if

$$\mathscr{L}_{(0)} = 1 = \frac{q^{\frac{1}{2}}}{1-q} = \mathscr{L}_{(1)},$$

whose solution is $\mathrm{F_C}(\zeta) = \frac{-1+\sqrt{5}}{2}$. Since s is infinite and $\mathscr{L}$ is vanishing by Claim 2 of Theorem 17.11, ζ has no other critical points. □

We recall the following parcel $\mathscr{Q}$, which appears in equation (1.4.4) by a different notation in the introduction.

Definition 17.17. *Let $s = (0, \infty)$, $l = 1$, $w = (1)$, and $\mathfrak{X} = \left\{q^{\frac{1}{2}}\right\}$. We define the $q^{\frac{1}{2}}$-quadratic monomial parcel*

$$\mathscr{Q} = \Lambda(s, l, w, \succ, \Psi_{s,\left(\left(\frac{1}{2},0,0\right)\right)}, x, \mathfrak{X}).$$

By quadratic primal monomial parcels and $\mathscr{Q}$, we have following analog of Corollary 17.16.

Corollary 17.18. *For the $q^{\frac{1}{2}}$-quadratic $\mathscr{Q} = \Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\rho,x,\mathfrak{X})$, suppose the ideal merged pair $\zeta = \Omega(\mathscr{Q})$.*

1. *For a quadratic primal monomial parcel $\mathscr{F} = \Lambda(s,l,w,\succ',\Psi_{s,\gamma',q},\rho',x,\mathfrak{X}')$, suppose the ideal merged pair $\zeta' = \Omega(\mathscr{F})$ with a front phase transition. Then, $\zeta \geq^x \zeta'$.*
2. *Consider all quadratic primal monomial parcels $\mathscr{F} = \Lambda(s,l,w,\succ',\Psi_{s,\gamma',q},\rho',x,\mathfrak{X}')$ such each $\Omega(\mathscr{F})$ has a front phase transition. Among them, $\mathscr{Q}$ gives the unique maximum ideal merged pair ζ with respect to the covering relation $\geq^x$.*
3. *The single critical point of ζ is the golden ratio $\mathrm{F_C}(\zeta) = \frac{-1+\sqrt{5}}{2}$.*

Proof. Claims 1, 2, and 3 hold as in Corollary 17.16, since $\mathscr{Q}_{(1)} = \frac{q^{\frac{1}{2}}}{1-q}$ and $\gamma'_{1,1} = \frac{1}{2}$ by the monomial conditions of (l,w,γ'). □

17.3 Convolutions of vanishing monomial parcels and phase transitions

Theorem 17.11 gives vanishing monomial parcels $\mathscr{F}$ with phase transitions. We obtain more vanishing parcels with phase transitions by convolutions. We first state the following, since convolutions of vanishing sequences are not necessarily vanishing.

Lemma 17.19. *Let $\delta \in \mathbb{Z}_{\geq 1}$. For each $i \in [\![\delta]\!]$, assume a primal monomial parcel $\mathscr{F}_i = \Lambda(s,l,w,\succ,\Psi_{s,\gamma_i,q},\rho,x,X)$ such that $\Omega(\mathscr{F}_i)$ is vanishing. Consider the parcel convolution $\mathscr{H} = *_{i\in[\![\delta]\!]}\mathscr{F}_i$. Then, $\Omega(\mathscr{H})$ is vanishing.*

Proof. Let $r \in \mathrm{O}_{\mathfrak{X}}$ and $\mu \in \mathbb{Z}_{\geq 0}$. Then, Proposition 17.6 gives $N_i(q(r)) \geq 1$ and $0 < S_i(q(r)) < 1$ such that $\mathscr{F}_{(\mu)}(r) \leq N_i(q(r)) \cdot S_i(q(r))^{\mu}$ for each $i \in [\![\delta]\!]$. Consider real numbers $\tilde{N}(q(r))$ and $\tilde{S}(q(r))$ such that each $i \in [\![\delta]\!]$ satisfies

$$\tilde{N}(q(r)) \geq N_i(q(r)),$$
$$1 > \tilde{S}(q(r)) \geq S_i(q(r)) > 0.$$

The assertion holds by $\mathscr{H}_{(\mu)}(r) \leq \tilde{N}(q(r))^{\delta} \cdot (\mu+1)^{\delta-1} \cdot \tilde{S}(q(r))^{\mu}$. □

Then, the following monomial indices give convolutions with front critical points.

Proposition 17.20. *Suppose integers $0 \leq d_1 \leq d_2$ such that $d_2 \in \mathbb{Z}_{\geq 1}$. Let $l = 1$ and $w = (1)$. Consider monomial indices (l,w,γ_i) for $i \in [\![d_2]\!]$ such that $i \in [\![d_1]\!]$ gives $\gamma_i = (0,\gamma_{i,2},0)$ with*

$$\gamma_{i,2} > 0, \tag{17.3.1}$$

and $i \in [\![d_1+1,d_2]\!]$ gives $\gamma_i = \left(\frac{1}{2},\gamma_{i,2},0\right)$ with

$$\gamma_{i,2} > -\frac{1}{2}. \tag{17.3.2}$$

For $i \in [\![d_2]\!]$, let $\mathscr{F}_i = \Lambda(s,l,w,\succ,\Psi_{s,\gamma_i,q},\rho,x,\mathfrak{X})$ be primal monomial parcels such that $\mathfrak{X} = \{q^{\kappa}\}$ of some $\kappa^{-1} \in \mathbb{Z}_{\geq 1}$. Moreover, consider the parcel convolution

$$\mathscr{H} = *_{i\in[\![d_2]\!]}\mathscr{F}_i = \Lambda(s,l,w,\succ,h_s,\rho,x,\mathfrak{X})$$

and the merged pair $\zeta = \Omega(\mathscr{H})$. Then, we have the following.

1. *ζ has the unique phase transition at the front critical point* $0 < \mathrm{F_C}(\zeta) = q^{\kappa} < 1$ *that solves*

$$1 - q = \sum_{i \in [\![d_1]\!]} q^{\gamma_{i,2}} + \sum_{i \in [\![d_1+1, d_2]\!]} q^{\frac{1}{2}+\gamma_{i,2}}.$$

2. *ζ has neither rear nor front critical points if one of inequalities* (17.3.1) *and* (17.3.2) *fails.*

Proof. Proof of Claim 1. Since s is infinite, ζ has no rear critical points. Moreover, each $\mathscr{F}_i$ satisfies Statement 2c of Theorem 17.11. Hence, ζ has no asymptotic critical points by Claim 2of Theorem 17.11 and Lemma 17.19.

Let us prove that ζ has the single front critical point. First, $p_1(q^{\kappa}) = 1-q$ is strictly decreasing over $0 \leq q^{\kappa} \leq 1$ with $p_1(0) = 1$ and $p_1(1) = 0$. Second, inequalities (17.3.1) and (17.3.2) imply that $p_2(q^{\kappa}) = \sum_{i \in [\![d_1]\!]} q^{\gamma_{i,2}} + \sum_{i \in [\![d_1+1, d_2]\!]} q^{\frac{1}{2}+\gamma_{i,2}}$ is strictly increasing over $0 \leq q^{\kappa} \leq 1$ with $p_2(0) = 0$ and $p_2(1) = d_2 \geq 1$.

Hence, there is the unique solution $0 < \mathrm{F_C}(\zeta) = q^{\kappa} < 1$ for

$$1 = \mathscr{H}_{(0)} = \mathscr{H}_{(1)} = \frac{p_2(q^{\kappa})}{p_1(q^{\kappa})}$$

such that $p_2(\mathrm{F_C}(\zeta)) = p_1(\mathrm{F_C}(\zeta))$. Also, there are $r_1, r_2 \in \mathrm{O}_{\mathfrak{X}}$ such that $\mathscr{H}_{(0)}(r_1) < \mathscr{H}_{(1)}(r_1)$ and $\mathscr{H}_{(0)}(r_2) > \mathscr{H}_{(1)}(r_2)$.

Proof of Claim2. If one of inequalities (17.3.1) and (17.3.2) fails, then we have no front critical points as $p_2(q^{\kappa}) > 1$ for $0 < q^{\kappa} < 1$. We have no rear critical points either for the infinite s. □

In particular, Proposition 17.20 determines front phase transitions of $\Omega(\mathscr{H})$ for each convolution $\mathscr{H}$ of primal monomial parcels. Moreover, the following gives explicit front critical points by metallic ratios.

Corollary 17.21. *Consider the $q^{\frac{1}{2}}$-linear $\mathscr{L} = \Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\rho,x,\mathfrak{X})$ and $q^{\frac{1}{2}}$-quadratic $\mathscr{Q} = \Lambda(s,l,w,\succ,\Psi_{s,\gamma',q},\rho,x,\mathfrak{X})$ of $\mathfrak{X} = \left\{q^{\frac{1}{2}}\right\}$. Let $n \in \mathbb{Z}_{\geq 1}$. Suppose merged pairs*

$$\zeta_n = \Omega(\mathscr{L}^{*n}),$$
$$\tilde{\zeta}_n = \Omega(\mathscr{Q}^{*n}).$$

Then, we have the following.

1. *$\mathfrak{X}$ is fully optimal for $\mathscr{L}^{*n}$ and $\mathscr{Q}^{*n}$.*

2. *ζ_n is ideal and has the unique phase transition at the front critical point* $\mathrm{F_C}(\zeta_n) = \frac{-n+\sqrt{n^2+4}}{2}$*, which is a metallic ratio.*

3. *The same holds for $\tilde{\zeta}_n$ with the front critical point* $\mathrm{F_C}(\tilde{\zeta}_n) = \mathrm{F_C}(\zeta_n)$.

Proof. Proof of Claim 1. Claim 1 holds by $\mathscr{L}^{*n}_{(1)} = \frac{nq^{\frac{1}{2}}}{1-q}$.

Proof of Claim 2. By Corollary 15.27 and Proposition 17.12, ζ_n is ideal. Thus, Claim 2 follows, since ζ_n has the front phase transition by the critical point $\mathrm{F_C}(\zeta_n)$ by Proposition 17.20.

Proof of Claims 3. Claims 3 holds similarly, since $\mathscr{Q}^{*n}_{(1)} = \mathscr{L}^{*n}_{(1)}$. □

18 Monomial convolutions and graded monomial products

We discuss graded monomial products and monomial convolutions in Section 1.10.

18.1 Merged-log-concavity of graded monomial products

We first prove that all monomial convolutions are generating functions of merged-log-concave parcels by parcel convolutions. Second, we prove the same for all graded monomial products by separable products.

For a multimonomial index $(d,w,\alpha,\beta,\gamma)$, the monomial convolution involves the change of variable $q \mapsto q^{\alpha_\lambda}$. Hence, we discuss the following change of parcel parameters.

Proposition 18.1. *Let $l \in \mathbb{Z}_{\geq 1}$ and $\kappa, \rho_1, \rho_2 \in \mathbb{Z}^l_{\geq 1}$. Consider a parcel $\mathscr{F} = \Lambda(s,l,w,\succ_1,f_s,\phi,\rho_1,x_1,\mathfrak{X})$ and $x_2 = x_1^{\kappa}$. Assume the following.*

1. *There are squaring orders $O_2 = \{\succeq_2, \succ_2\} \supseteq O_1 = \{\succeq_1, \succ_1\}$.*
2. *ϕ is a $(s,l,w,\succ_2,\kappa,x_1,\mathfrak{X})$-mediator.*
3. *ϕ is a $(s,l,w,\succ_2,\rho_2,x_2,\mathfrak{X})$-mediator.*

Then, we have the following.

(a) There exists a parcel $\mathscr{G} = \Lambda(s,l,w,\succ_2,g_s,\phi,\rho_2,x_2,\mathfrak{X}) = \mathscr{F}$ such that each $m \in [\![s]\!]^l$ satisfies

$$g_{s,m} = f_m \cdot B(s,l,w,m,\phi,\kappa,x_1,\mathfrak{X}).$$

(b) Suppose squaring orders $O' = \{\succeq', \succ'\} \supseteq O_2$. For $i \in [\![2]\!]$, let

$$\mu_{\succ',i} = \left(s,l,w,\succ',\phi,\rho_i,x_i,\mathfrak{X}\right),$$
$$\mu_{\succeq',i} = \left(s,l,w,\succeq',\phi,\rho_i,x_i,\mathfrak{X}\right).$$

Furthermore, let $\kappa \circ \rho_2 = \rho_1$. Then, $\mathscr{F}$ is $\mu_{\succ',2}$-merged-log-concave if and only if $\mathscr{F}$ is $\mu_{\succ',1}$-merged-log-concave. Also, $\mathscr{F}$ is $\mu_{\succeq',2}$-merged-log-concave if and only if $\mathscr{F}$ is $\mu_{\succeq',1}$-merged-log-concave.

Proof. Proof of Claim (a). For each $m \in [\![s]\!]^l$,

$$\mathscr{F}_m = \frac{f_{s,m}}{\prod \phi(x_2)^{m \circ w} \cdot [m]!^w_{x_2}} \frac{\prod \phi(x_2)^{m \circ w} \cdot [m]!^w_{x_2}}{\prod \phi(x_1)^{m \circ w} \cdot [m]!^w_{x_1}} = \frac{g_{s,m}}{\prod \phi(x_2)^{m \circ w} \cdot [m]!^w_{x_2}}.$$

Also, g_s is $\succ_2$-positive by Assumptions 1 and 2, since f_s is $\succ_1$-positive. Hence, Assumption 3 gives Claim (a).

Proof of Claim (b). Claim (b) holds by Claim (a), because $x_2^{\rho_2} = x_1^{\rho_1}$ implies

$$\Upsilon(s,l,w,m,n,k,\phi,\rho_1,x_1,\mathfrak{X}) = \Upsilon(s,l,w,m,n,k,\phi,\rho_2,x_2,\mathfrak{X})$$

for each $m,n \in \mathbb{Z}^l$ and $k \in \mathbb{Z}^{2l}$. □

We introduce the following notations of multimonomial indices by Definitions 1.11 and 1.13.

Definition 18.2. *For a reduced multimonomial index (d,w,α,γ) and $\beta = \iota^d(1)$, let*

$$M(d,w,\alpha,\gamma,q,z) = M(d,w,\alpha,\beta,\gamma,q,z),$$
$$\mathscr{M}(d,w,\alpha,\gamma,q,v) = \mathscr{M}(d,w,\alpha,\beta,\gamma,q,v).$$

We call $M(d,w,\alpha,\gamma,q,z)$ and $\mathscr{M}(d,w,\alpha,\gamma,q,v)$ a reduced graded monomial product and reduced monomial convolution (or a graded monomial product and monomial convolution for short).

In particular, for each monomial convolution $\mathscr{M}(d,w,\alpha,\beta,\gamma,q,v)$, there exists a reduced monomial convolution $\mathscr{M}(d',w,\alpha',\gamma',q,v) = \mathscr{M}(d,w,\alpha,\beta,\gamma,q,v)$.

We now obtain the merged-log-concavity of all monomial convolutions. For parcels $\mathscr{F}_1,\ldots,\mathscr{F}_d$, we write $*_{i\in[\![d]\!]}\mathscr{F}_i$ for the parcel convolution $\mathscr{F}_1 * \cdots * \mathscr{F}_d$.

Theorem 18.3. *Consider a multimonomial index $(d,w,\alpha,\beta,\gamma)$ with $\delta \in \mathbb{Z}_{\geq 1}$ and $\kappa \in \mathbb{Z}_{\geq 1}^d$ such that $\kappa \circ \alpha = \iota^d(\delta)$. Let $s = (0,\infty)$, $l = 1$, $\succ = >_{\alpha,\gamma,q}$, $\rho \in \mathbb{Z}_{\geq 1}^l$, $y = \left(q^{\delta}\right)$, and $\mathfrak{X} = \mathfrak{U}_{\alpha,\gamma,q}$. For $i \in [\![d]\!]$, let $x_i = (q^{\alpha_i})$ and*

$$\mathscr{F}_i = \Lambda(s,l,w,\succ,\Psi_{s,(\gamma_i),q^{\alpha_i}},p_i,x_i,\mathfrak{X}).$$

Then, we have the following.

1. *There is the $\succ$-merged-log-concave parcel*

$$\mathscr{G} = \Lambda(s,l,w,\succ,f_s,\rho,y,\mathfrak{X}) = *_{i\in[\![d]\!]}\mathscr{F}_i^{*\beta_i}.$$

2. *For an indeterminate v,*

$$\mathscr{Z}_{\mathscr{G}}(v) = \mathscr{M}(d,w,\alpha,\beta,\gamma,q,v).$$

Proof. Proof of Claim 1. Without loss of generality, assume a reduced multimonomial index (d,w,α,γ). First, let $d = 1$. Then, we have $y = x_1^{\kappa}$ and $\mathscr{F}_1 = \Lambda(s,l,w,\succ,\Psi_{s,(\gamma_1),q^{\alpha_1}},\phi,p_1,x_1,\mathfrak{X})$ for the l-canonical mediator ϕ. Hence, Claim (a) of Proposition 18.1 gives the parcel

$$\mathscr{G} = \Lambda(s,l,w,\succ,f_s,\rho,y,\mathfrak{X}) = \mathscr{F}_1.$$

By Theorem 8.40, $\mathscr{G}$ is $(s,l,w,\succ,\phi,p_1,x_1,\mathfrak{X})$-merged-log-concave. Hence, Claim 1 holds, since $\mathscr{G}$ is $(s,l,w,\succ,\phi,\rho,y,\mathfrak{X})$-merged-log-concave by Claim (b) of Proposition 18.1.

Second, let $d \geq 2$. Then, the induction gives the $\succ$-merged-log-concave parcel

$$\mathscr{H} = \Lambda(s,l,w,\succ,h_s,\rho,y,\mathfrak{X}) = *_{i\in[\![d-1]\!]}\mathscr{F}_i.$$

Also, we have $\mathscr{F}_d = \Lambda(s,l,w,\succ,\Psi_{s,(\gamma_i),q^{\alpha_i}},\phi,\kappa_d\rho,x_i,\mathfrak{X})$ for the canonical l-mediator ϕ. Let $\lambda = (\delta,\alpha_d,\delta)$, $\tau = (\rho,\kappa_d\rho,\rho)$, $O = (O_i = \{\succeq,\succ\})_{i\in[\![3]\!]}$, and $o = (1,\kappa_d)$. Hence, $(\mathscr{H},\mathscr{F}_d)$ carries the convolution index $\left(\iota^3(s),l,w,O,\phi,\tau,(y,x_d,y),\mathfrak{X},q,\lambda,o\right)$ by the exponent equation $o_1\lambda_1 = o_2\lambda_2 = \delta = \lambda_3$, the base-shift equation $o_1^{-1}\tau_1 = o_2^{-1}\tau_2 = \rho = \tau_3$, and Claim 4 of Lemma 15.2. Theorem 15.25 now gives the $\succ$-merged-log-concave parcel

$$\mathscr{G} = \Lambda(s,l,w,\succ,f_s,\rho,y,\mathfrak{X}) = \mathscr{H} * \mathscr{F}_d = *_{i\in[\![d]\!]}\mathscr{F}_i.$$

Proof of Claim 2. Claim 2 follows from Claim 1. □

We introduce the following notation, which generalizes Definition 1.19.

Definition 18.4. *Let $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\phi,\rho,x,\mathfrak{X})$ and $z = (z_i)_{i\in[\![l]\!]}$ be indeterminates z_i.*

1. *We write the generating function $\mathscr{Z}_{\mathscr{F}}(z) = \mathscr{Z}_{\mathscr{F}}(z_1,\ldots,z_l)$ of $\mathscr{F}$ such that*

$$\mathscr{Z}_{\mathscr{F}}(z) = \sum_{\lambda\in\mathbb{Z}_{\geq 0}^l} \mathscr{F}_\lambda \cdot \prod_{i\in[\![l]\!]} z_i^{\lambda_i} \in \mathbb{Q}(\mathfrak{X})[[z_1,\ldots,z_l]].$$

2. *In particular, if $s=(0,\infty)$ and $\mathscr{F}=\Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\rho,x,\mathfrak{X})$, then let*

$$\mathscr{Z}_{w,\gamma,q}(z)=\mathscr{Z}_{\mathscr{F}}(z_1,\ldots,z_l).$$

When $w=\iota^l(1)$, let $\mathscr{Z}_{\gamma,q}(z)=\mathscr{Z}_{w,\gamma,q}(z)$.

Hence, we obtain the following merged-log-concavity of graded monomial products.

Theorem 18.5. *Consider a graded monomial product $M(d,w,\alpha,\beta,\gamma,q,z)$. Let $s=(0,\infty)$, $\succ=>_{\alpha,\gamma,q}$, $\rho\in\mathbb{Z}^d_{\geq 1}$, $x=(q^{\alpha_i})_{i\in[\![d]\!]}$, and $\mathfrak{X}=\mathfrak{U}_{\alpha,\gamma,q}$. Also, suppose $z=(z_i)_d$ of indeterminates z_i. Then, we have the merged-log-concave parcel $\mathscr{F}=\Lambda(s,d,w,\succ,f_s,\rho,x,\mathfrak{X})$ such that*

$$\mathscr{Z}_{\mathscr{F}}(z)=M(d,w,\alpha,\beta,\gamma,q,z). \tag{18.1.1}$$

Proof. For each $i\in[\![d]\!]$, Theorem 18.3 gives the merged-log-concave $\mathscr{H}_i=\Lambda(s,1,w,\succ,f_{i,s},\rho,x_i,\mathfrak{X})$ such that

$$\mathscr{M}(1,w,(\alpha_i),(\beta_i),(\gamma_i),q,z_i)=\mathscr{Z}_{\mathscr{H}_i}(z_i).$$

Then, Theorem 9.7 yields the merged-log-concave

$$\mathscr{F}=\Lambda(s,d,w,\succ,f_s,\rho,x,\mathfrak{X})=\mathscr{H}_1\diamond\ldots\diamond\mathscr{H}_d$$

in equation (18.1.1). □

18.2 Graded monomial products and eta products

We first realize the eta function $\eta(\tau)$ in Definition 1.22 and its inverse by weight-one linear and quadratic monomial parcels.

Lemma 18.6. *Let $s=(0,\infty)$, $l=1$, and $w=(1)$. Consider $\kappa_1,\kappa_2\in\mathbb{Q}$ such that $\kappa_1+\kappa_2=1$. Let*

$$\mathscr{F}_1=\Lambda(s,l,w,\succ,\Psi_{s,\left(\left(0,\kappa_1,-\frac{1}{24}\right)\right),q},\rho,x,\mathfrak{X}),$$
$$\mathscr{F}_2=\Lambda(s,l,w,\succ,\Psi_{s,\left(\left(\frac{1}{2},\frac{\kappa_1}{2},\frac{1}{24}\right)\right),q},\rho,x,\mathfrak{X}).$$

Then, $q=e^{2\pi i\tau}$ of $\operatorname{Im}\tau>0$ satisfies

$$\mathscr{Z}_{\mathscr{F}_1}(q^{\kappa_2})=\eta(\tau)^{-1},$$
$$\mathscr{Z}_{\mathscr{F}_2}(-q^{\frac{\kappa_2}{2}})=\eta(\tau).$$

Proof. Since $\eta(\tau)=q^{\frac{1}{24}}(q;q)$, the Euler binomial identities imply

$$\begin{aligned}\eta(\tau)^{-1}&=q^{-\frac{1}{24}}\sum_{\lambda\in\mathbb{Z}_{\geq 0}}\frac{q^{\lambda}}{(\lambda)_q}\\&=q^{-\frac{1}{24}}\sum_{\lambda\in\mathbb{Z}_{\geq 0}}\frac{q^{\lambda(\kappa_1+\kappa_2)}}{(\lambda)_q}\\&=q^{-\frac{1}{24}}\sum_{\lambda\in\mathbb{Z}_{\geq 0}}\frac{q^{\lambda\kappa_1}}{(\lambda)_q}(q^{\kappa_2})^{\lambda}\\&=\mathscr{Z}_{\mathscr{F}_2}(q^{\kappa_2}),\end{aligned}$$

$$
\begin{aligned}
\eta(\tau) &= q^{\frac{1}{24}} \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{q^{\frac{\lambda(\lambda-1)}{2}}}{(\lambda)_q} (-q)^{\lambda} \\
&= q^{\frac{1}{24}} \sum_{\lambda \in \mathbb{Z}_{\geq 0}} (-1)^{\lambda} \frac{q^{\frac{\lambda(\lambda+1)}{2}}}{(\lambda)_q} \\
&= q^{\frac{1}{24}} \sum_{\lambda \in \mathbb{Z}_{\geq 0}} (-1)^{\lambda} \frac{q^{\frac{\lambda^2}{2} + \lambda \frac{\kappa_1+\kappa_2}{2}}}{(\lambda)_q} \\
&= q^{\frac{1}{24}} \sum_{\lambda \in \mathbb{Z}_{\geq 0}} \frac{q^{\frac{\lambda^2}{2} + \lambda \frac{\kappa_1}{2}}}{(\lambda)_q} (-q^{\frac{\kappa_2}{2}})^{\lambda} \\
&= \mathscr{Z}_{\mathscr{F}_1}(-q^{\frac{\kappa_2}{2}}).
\end{aligned}
$$

□

Hence, we have the following by $\gamma(\beta,\kappa)$ and $T(z,q,\alpha,\beta,\kappa)$ in Definition 1.24. Then, graded monomial products $M(d,w,\alpha,|\beta|,\gamma(\beta,\kappa),q,z)$ give the merged-log-concavity of infinitely-many $T(z,q,\alpha,\beta,\kappa)$-analogs of each eta product $E_{d,\alpha,\beta}(\tau)$ by choices of $\kappa \in \mathbb{Q}^d$.

Proposition 18.7. *Let $d \in \mathbb{Z}_{\geq 1}$, $w = (1)$, $\alpha \in \mathbb{Z}^d_{\geq 1}$, $\beta \in \mathbb{Z}^d_{\neq 0}$, and $\kappa \in \mathbb{Q}^d$. Then, we have the following.*

1. *There exists a width-d merged-log-concave parcel $\mathscr{F}$ such that*

$$\mathscr{Z}_{\mathscr{F}}(z) = M(d,w,\alpha,|\beta|,\gamma(\beta,\kappa),q,z).$$

2. *For each $q = e^{2\pi \iota \tau}$ of $\operatorname{Im}(\tau) > 0$,*

$$E_{d,\alpha,\beta}(\tau) = \lim_{T(z,q,\alpha,\beta,\kappa) \to \iota^d(1) \in \mathbb{C}^d} \mathscr{Z}_{\mathscr{F}}(z).$$

Namely, $\mathscr{Z}_{\mathscr{F}}(z)$ is a t-analog of $E_{d,\alpha,\beta}(\tau)$ for $t = T(z,q,\alpha,\beta,\kappa)$.

Proof. Proof of Claim 1. Claim 1 follows from Theorem 18.5, since $(1,w,(\gamma(\beta,\kappa)_i))$ is a monomial index for each $i \in [\![d]\!]$.

Proof of Claim 2. Let $\alpha_i = 1$ for simplicity. First, assume $\beta_i \geq 1$ for some $i \in [\![d]\!]$. Then, when $T(z,q,\alpha,\beta,\kappa) \mapsto \iota^d(1) \in \mathbb{C}^d$ as z varies, we have

$$-q^{-\frac{1-\kappa_i}{2}} \cdot z_i \to 1,$$

which implies

$$\left| q^{\frac{\kappa_i}{2}} z_i \right| \to \left| -q^{\frac{1}{2}} \right| < 1.$$

Hence, suppose $z_i \in \mathbb{C}$ such that $-q^{-\frac{1-\kappa_i}{2}} \cdot z_i$ is sufficiently close to 1 so that

$$\left| q^{\frac{\kappa_i}{2}} z_i \right| < 1. \tag{18.2.1}$$

Also, let $a_\lambda(q,z_i)=\frac{q^{\frac{\lambda^2}{2}+\frac{\kappa_i\lambda}{2}}}{(\lambda)_q}z_i^\lambda$. Then, we have

$$\left|a_\lambda(q,z_i)z_i^\lambda\right|=\left|\frac{q^{\frac{\lambda^2}{2}}}{(\lambda)_q}(q^{\frac{\kappa_i}{2}}z_i)^\lambda\right|<\left|\frac{q^{\frac{\lambda^2}{2}}}{(\lambda)_q}\right|$$

Consider a primal monomial parcel $\mathscr{G}=\Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\rho,x,\mathfrak{X})$ such that $\mathscr{G}_m=\frac{q^{\frac{\lambda^2}{2}}}{(\lambda)_q}$. Then, $\mathscr{G}$ is probabilistic by Proposition 17.6. This implies that $\sum_{\lambda\in\mathbb{Z}_{\geq 0}}|a_\lambda(q,z_i)|$ converges absolutely and uniformly by the Weierstrass M-test for all $z_i\in\mathbb{C}$ that satisfy inequality (18.2.1). Therefore, the dominated convergence theorem gives

$$\begin{aligned}\lim_{T(z,q,\alpha,\beta,\kappa)\mapsto\iota^d(1)\in\mathbb{C}^d}q^{\frac{1}{24}}\sum_{\lambda\in\mathbb{Z}_{\geq 0}}a_\lambda(q,z_i)&=q^{\frac{1}{24}}\sum_{\lambda\in\mathbb{Z}_{\geq 0}}\lim_{T(z,q,\alpha,\beta,\kappa)\mapsto\iota^d(1)\in\mathbb{C}^d}a_\lambda(q,z_i)\\&=q^{\frac{1}{24}}\sum_{\lambda\in\mathbb{Z}_{\geq 0}}\frac{q^{\frac{\lambda^2}{2}}}{(\lambda)_q}(-q^{\frac{1}{2}})^\lambda\\&=\eta(\tau).\end{aligned}$$

Second, assume $\beta_i\leq -1$ for some $i\in[\![d]\!]$. This case holds similarly as follows. When $T(z,q,\alpha,\beta,\kappa)\mapsto\iota^d(1)\in\mathbb{C}^d$ as z varies, we have

$$q^{-(1-\kappa_i)}\cdot z_i\to 1,$$

which implies

$$|q^{\kappa_i}z_i|\to|q|<\left|q^{\frac{1}{2}}\right|<1.$$

Now, suppose that $q^{-(1-\kappa_i)}\cdot z_i$ is sufficiently close to 1 so that

$$|q^{\kappa_i}z_i|<\left|q^{\frac{1}{2}}\right|. \tag{18.2.2}$$

Also, let $a_\lambda(q,z_i)=\frac{q^{\lambda\kappa_i}}{(\lambda)_q}z_i^\lambda$. Then, we have

$$|a_\lambda(q,z_i)|=\left|\frac{(q^{\kappa_i}z_i)^\lambda}{(\lambda)_q}\right|<\left|\frac{q^{\frac{\lambda}{2}}}{(\lambda)_q}\right|$$

Consider a primal monomial parcel $\mathscr{G}=\Lambda(s,l,w,\succ,\Psi_{s,\gamma,q},\rho,x,\mathfrak{X})$ such that $\mathscr{G}_m=\frac{q^{\frac{\lambda}{2}}}{(\lambda)_q}$. Then, $\mathscr{G}$ is probabilistic by Proposition 17.6. Hence, $\sum_{\lambda\in\mathbb{Z}_{\geq 0}}|a_\lambda(q,z_i)|$ converges absolutely and uniformly by the Weierstrass M-test for all $z_i\in\mathbb{C}$ that satisfy inequality (18.2.2). In particular, the dominated convergence theorem gives

$$\begin{aligned}\lim_{T(z,q,\alpha,\beta,\kappa)\mapsto\iota^d(1)\in\mathbb{C}^d}q^{-\frac{1}{24}}\sum_{\lambda\in\mathbb{Z}_{\geq 0}}a_\lambda(q,z_i)&=q^{-\frac{1}{24}}\sum_{\lambda\in\mathbb{Z}_{\geq 0}}\lim_{T(z,q,\alpha,\beta,\kappa)\mapsto\iota^d(1)\in\mathbb{C}^d}a_\lambda(q,z_i)\\&=q^{-\frac{1}{24}}\sum_{\lambda\in\mathbb{Z}_{\geq 0}}\frac{q^\lambda}{(\lambda)_q}\\&=\eta(\tau)^{-1}.\end{aligned}$$

□

Example 18.8. Let $s=(0,\infty)$, $d=3$, $w=(1)$, $\alpha_1=(1,1,1)$, $\alpha_2=(2,1,1)$, $\beta=(1,2,2)$, $\rho=\iota^d(1)$, and $\gamma=\left((0,0,0),\left(0,\frac{1}{2},0\right),(0,0,0)\right)$. Also, let $x_i=\iota^d(q)^{\alpha_i}$, $\succ_i=>_{\alpha_i,\gamma,q}$, and $\mathfrak{X}_i=\mathfrak{U}_{\alpha_i,\gamma,q}$ for $i\in[\![2]\!]$. Then, Claim 1 in Proposition 18.7 gives width-d parcels $\mathscr{F}_i=\Lambda(s,d,w,\succ_i,f_s,\rho,x_i,\mathfrak{X}_i)$ such that each $i\in[\![2]\!]$ satisfies

$$\mathscr{Z}_{\mathscr{F}_i}(z_1,\ldots,z_d)=M(d,w,\alpha_i,\beta,\gamma,q,z).$$

Hence, suppose $\mathbb{P}_{s,d,\xi,h}=\left(\mathbb{P}_{s,d,\xi,h,i}=(s,d,m_i,n_i,k_i)\right)_{i\in[\![\theta]\!]}$ in Example 12.34 with $k_i=(0,1,0,2,1,0)$. For instance, $\mathscr{F}_{1,m_1}=\frac{4q^{\frac{1}{2}}}{(q^2-1)(q^4-1)(q-1)^2}$. Moreover, we have the following unimodal q-polynomial:

$$\begin{aligned}\Delta(\mathscr{F}_1)(s,d,w,m_1,n_1,k_1,\rho,x_1,\mathfrak{X}_1)&=q^{38}+2q^{37}+7q^{36}+12q^{35}+25q^{34}+38q^{33}+63q^{32}+88q^{31}\\&+128q^{30}+168q^{29}+221q^{28}+274q^{27}+331q^{26}+388q^{25}\\&+437q^{24}+486q^{23}+515q^{22}+544q^{21}+544q^{20}+544q^{19}+515q^{18}\\&+486q^{17}+437q^{16}+388q^{15}+331q^{14}+274q^{13}+221q^{12}+168q^{11}\\&+128q^{10}+88q^{9}+63q^{8}+38q^{7}+25q^{6}+12q^{5}+7q^{4}+2q^{3}+q^{2}.\end{aligned}$$

However, the following is not a unimodal q-polynomial:

$$\begin{aligned}\Delta(\mathscr{F}_2)(s,d,w,m_1,n_1,k_1,\rho,x_2,\mathfrak{X}_2)&=q^{30}+q^{29}+5q^{28}+5q^{27}+14q^{26}+13q^{25}+29q^{24}+25q^{23}\\&+49q^{22}+40q^{21}+70q^{20}+54q^{19}+86q^{18}+62q^{17}\\&+92q^{16}+62q^{15}+86q^{14}+54q^{13}+70q^{12}+40q^{11}+49q^{10}\\&+25q^{9}+29q^{8}+13q^{7}+14q^{6}+5q^{5}+5q^{4}+q^{3}+q^{2}.\end{aligned}$$

Since we are interested in the unimodality, we conjecture the following (see Conjecture 8.59 for $M(1,(1),(1),(1),((0,0,0)),q,(z_1)))$.

Conjecture 18.9. *Under the notation of Example 18.8, $\Delta(\mathscr{F}_1)(s,d,w,m_i,n_i,k_i,\rho,x_1,\mathfrak{X}_1)$ is a unimodal q-polynomial for each $i\in\mathbb{Z}_{\geq 1}$.*

18.3 Weighted q–multinomial coefficients and monomial convolutions

We employ the following notation.

Definition 18.10. *For $d\in\mathbb{Z}_{\geq 1}$, $\lambda\in[\![d]\!]$, and $j\in\mathbb{Z}^d$, let*

$$o(j,\lambda)=(j_1,\ldots,j_\lambda-1,\ldots,j_d)\in\mathbb{Z}^d.$$

For instance, $j\in\mathbb{Z}^d$ and $i\in\mathbb{Z}_{\geq 1}$ give the q-Pascal identity $\begin{bmatrix}i\\j\end{bmatrix}_q=\sum_{\lambda\in[\![d]\!]}q^{\sum_{\kappa\in[\![\lambda-1]\!]}j_\kappa}\begin{bmatrix}i-1\\o(j,\lambda)\end{bmatrix}_q$. We extend this by weighted q-multinomial coefficients.

Proposition 18.11. *Let $d\in\mathbb{Z}_{\geq 1}$, $\alpha\in\mathbb{Z}^d_{\geq 1}$, $\delta_l=\operatorname{lcm}(\alpha)$, and $\delta_g=\gcd(\alpha)$. If $i\in\mathbb{Z}_{\geq 1}$ and $j\in\mathbb{Z}^d$, then we have a weighted q-Pascal identity*

$$\begin{bmatrix}i\\j\end{bmatrix}_{\alpha,q}=\sum_{\lambda\in[\![d]\!]}q^{\delta_l\sum_{\kappa\in[\![\lambda-1]\!]}j_\kappa}\begin{bmatrix}i-1\\o(j,\lambda)\end{bmatrix}_{\alpha,q}\begin{bmatrix}\frac{\delta_l}{\alpha_\lambda}\end{bmatrix}_{q^{\alpha_\lambda j_\lambda}}.$$

In particular, $\begin{bmatrix}i\\j\end{bmatrix}_{\alpha,q}\geq_{q^{\delta_g}}0$.

Proof. Assume $j \in \mathbb{Z}^d_{\geq 0}$ and $\sum j = i$; otherwise, the weighted q-Pascal identity holds by $0 = 0$. Then, by the unweighted q-Pascal identity, we have

$$\begin{aligned}\begin{bmatrix} i \\ j \end{bmatrix}_{\alpha,q} &= \begin{bmatrix} i \\ j \end{bmatrix}_{q^{\delta_l}} \cdot \frac{\prod_{\lambda \in [\![d]\!]} (j_\lambda)_{q^{\delta_l}}}{\prod_{\lambda \in [\![d]\!]} (j_\lambda)_{q^{\alpha_\lambda}}} \\ &= \sum_{\lambda \in [\![d]\!]} q^{\delta_l \sum_{\kappa \in [\![\lambda-1]\!]} j_\kappa} \begin{bmatrix} i-1 \\ o(j,\lambda) \end{bmatrix}_{q^{\delta_l}} \cdot \frac{\prod_{\lambda \in [\![d]\!]} (j_\lambda)_{q^{\delta_l}}}{\prod_{\lambda \in [\![d]\!]} (j_\lambda)_{q^{\alpha_\lambda}}}.\end{aligned}$$

On each summand above, if $j_\lambda < 1$, then $\left[\begin{smallmatrix} i-1 \\ o(j,\lambda) \end{smallmatrix}\right]_{q^{\delta_l}} = 0 = \left[\begin{smallmatrix} i-1 \\ o(j,\lambda) \end{smallmatrix}\right]_{\alpha,q}$. If not, then since each α_λ divides δ_l, we have

$$\begin{aligned}\begin{bmatrix} i-1 \\ o(j,\lambda) \end{bmatrix}_{q^{\delta_l}} \cdot \frac{\prod_{\lambda \in [\![d]\!]} (j_\lambda)_{q^{\delta_l}}}{\prod_{\lambda \in [\![d]\!]} (j_\lambda)_{q^{\alpha_\lambda}}} &= \frac{(i-1)_{q^{\delta_l}}}{\prod_{\kappa \in [\![d]\!]} (o(j,\lambda)_\kappa)_{q^{\delta_l}}} \cdot \frac{\prod_{\kappa \in [\![d]\!]} (j_\kappa)_{q^{\delta_l}}}{\prod_{\kappa \in [\![d]\!]} (j_\kappa)_{q^{\alpha_\kappa}}} \\ &= \begin{bmatrix} i-1 \\ o(j,\lambda) \end{bmatrix}_{\alpha,q} \cdot \frac{(1-q^{\delta_l j_\lambda})}{(1-q^{\alpha_\lambda j_\lambda})} \\ &= \begin{bmatrix} i-1 \\ o(j,\lambda) \end{bmatrix}_{\alpha,q} \cdot \begin{bmatrix} \frac{\delta_l}{\alpha_\lambda} \end{bmatrix}_{q^{\alpha_\lambda j_\lambda}}.\end{aligned}$$

Thus, we obtain the weighted q-Pascal identity.

Since $\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]_{\alpha,q} = 1$ or 0 when $i \in \mathbb{Z}_{\leq 0}$, the weighted q-Pascal identity gives the latter statement by the induction on $i \in \mathbb{Z}_{\geq 0}$. □

Example 18.12. Let $\alpha_\lambda = (1, \lambda, \lambda)$ for $\lambda \in [\![2]\!]$. Then, we have the following unimodal q-polynomials:

$$\begin{aligned}\begin{bmatrix} 3 \\ (1,1,1) \end{bmatrix}_{\alpha_1,q} &= \frac{(3)_q}{(1)_q(1)_q(1)_q} = q^3 + 2q^2 + 2q + 1; \\ \begin{bmatrix} 3 \\ (1,1,1) \end{bmatrix}_{\alpha_2,q} &= \frac{(3)_{q^2}}{(1)_q(1)_{q^2}(1)_{q^2}} = q^7 + q^6 + 2q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1.\end{aligned}$$

However, $\left[\begin{smallmatrix} 10 \\ (4,3,3) \end{smallmatrix}\right]_{\alpha_2,q}$ is not a unimodal q-polynomial by $\cdots + 2409q^{39} + 2390q^{38} + 2409q^{37} + \cdots$ in

$$\begin{aligned}&q^{76} + q^{75} + 3q^{74} + 4q^{73} + 9q^{72} + 11q^{71} + 21q^{70} + 26q^{69} + 43q^{68} + 53q^{67} + 80q^{66} \\ &+ 97q^{65} + 138q^{64} + 165q^{63} + 221q^{62} + 262q^{61} + 336q^{60} + 392q^{59} + 485q^{58} + 559q^{57} \\ &+ 668q^{56} + 761q^{55} + 884q^{54} + 993q^{53} + 1125q^{52} + 1248q^{51} + 1380q^{50} + 1512q^{49} \\ &+ 1637q^{48} + 1769q^{47} + 1879q^{46} + 2004q^{45} + 2088q^{44} + 2198q^{43} + 2252q^{42} \\ &+ 2336q^{41} + 2355q^{40} + 2409q^{39} + 2390q^{38} + 2409q^{37} + 2355q^{36} + 2336q^{35} \\ &+ 2252q^{34} + 2198q^{33} + 2088q^{32} + 2004q^{31} + 1879q^{30} + 1769q^{29} + 1637q^{28} \\ &+ 1512q^{27} + 1380q^{26} + 1248q^{25} + 1125q^{24} + 993q^{23} + 884q^{22} + 761q^{21} + 668q^{20} \\ &+ 559q^{19} + 485q^{18} + 392q^{17} + 336q^{16} + 262q^{15} + 221q^{14} + 165q^{13} + 138q^{12} \\ &+ 97q^{11} + 80q^{10} + 53q^9 + 43q^8 + 26q^7 + 21q^6 + 11q^5 + 9q^4 + 4q^3 + 3q^2 + q + 1.\end{aligned}$$

Thus, it would be interesting to clarify the unimodality of weighted q-multinomial coefficients, as that of non-weighted ones is important [Oha, Syl]. For instance, a computer program checks that $\left[\begin{smallmatrix} 30 \\ j \end{smallmatrix}\right]_{(1,4,8),q}$ is a unimodal q-polynomial for each $j \in \mathbb{Z}^3_{\geq 1}$.

18.4 Monomial convolutions and merged determinants

We write monomial convolutions by the weighted q-multinomial coefficients and the following rational functions by Definitions 1.13 and 1.14.

Proposition 18.13. *Suppose a multimonomial index (d,w,α,γ) with $\delta = \mathrm{lcm}(\alpha)$. Then, we have the following.*

1. $\mathscr{M}(d,w,\alpha,\gamma,q,v) = \sum_{i\in\mathbb{Z}_{\geq 0}} \frac{\sum_{j\in\mathbb{Z}^d} \psi_{\alpha,\gamma,q,j} \left[{i \atop j}\right]^{w_1}_{\alpha,q}}{(i)^{w_1}_{q^\delta}} \cdot v^i.$

2. *Let $s=(0,\infty)$, $l=1$, and $\rho \in \mathbb{Z}^l_{\geq 1}$. Then, there is the merged-log-concave parcel $\mathscr{F} = \Lambda(s,l,w,>_{\alpha,\gamma,q},f_s,\rho,\left(q^\delta\right),\mathfrak{U}_{\alpha,\gamma,q})$ such that $f_{s,m} = \sum_{j\in\mathbb{Z}^d} \psi_{\alpha,\gamma,q,j}\left[{m_1 \atop j}\right]^{w_1}_{\alpha,q}$ and*

$$\mathscr{Z}_{\mathscr{F}}(v) = \mathscr{M}(d,w,\alpha,\gamma,q,v).$$

Proof. Proof of Claim 1. We have $\mathscr{M}(d,w,\alpha,\gamma,q,v) = \prod_{\lambda\in[\![d]\!]}\left(\sum_{i\in\mathbb{Z}_{\geq 0}} \frac{\psi_{(\alpha_\lambda),(\gamma_\lambda),q,(i)}}{(i)^{w_1}_{q^{\alpha_\lambda}}}\cdot v^i\right)$. Thus, we obtain Claim 1, because

$$\begin{aligned}
\mathscr{M}(d,w,\alpha,\gamma,q,v) &= \sum_{i\in\mathbb{Z}_{\geq 0}} \left(\sum_{j\in\mathbb{Z}^d_{\geq 0},\sum j=i} \frac{\psi_{\alpha,\gamma,q,j}}{\prod_{\lambda\in[\![d]\!]}(j_\lambda)^{w_1}_{q^{\alpha_\lambda}}}\right) v^i \\
&= \sum_{i\in\mathbb{Z}_{\geq 0}} \left(\sum_{j\in\mathbb{Z}^d_{\geq 0},\sum j=i} \psi_{\alpha,\gamma,q,j}\cdot \frac{(i)^{w_1}_{q^\delta}}{\prod_{\lambda\in[\![d]\!]}(j_\lambda)^{w_1}_{q^{\alpha_\lambda}}}\cdot\frac{1}{(i)^{w_1}_{q^\delta}}\right) v^i \\
&= \sum_{i\in\mathbb{Z}_{\geq 0}} \frac{\sum_{j\in\mathbb{Z}^d_{\geq 0},\sum j=i}\psi_{\alpha,\gamma,q,j}\left[{i \atop j}\right]^{w_1}_{\alpha,q}}{(i)^{w_1}_{q^\delta}}\cdot v^i.
\end{aligned}$$

Proof of Claim 2. Claim 2 follows from Theorem 18.3 and Claim 1. □

We define the following merged determinants of monomial convolutions.

Definition 18.14. *Let $l=1$, $\rho\in\mathbb{Z}^l_{\geq 1}$, $m,n\in\mathbb{Z}^l$, and $k\in\mathbb{Z}^{2l}$. Consider a multimonomial index $(d,w,\alpha,\beta,\gamma)$ with $x = \left(q^{\mathrm{lcm}(\alpha)}\right)$ and $\mathfrak{X} = \mathfrak{U}_{\alpha,\gamma,q}$. Then, we define the merged determinant*

$$\begin{aligned}
&\Delta(d,w,\alpha,\beta,\gamma,m,n,k,\rho,q) \\
&= \Upsilon(s,l,w,m,n,k,\rho,x,\mathfrak{X})\det\begin{bmatrix} \mathscr{M}(d,w,\alpha,\beta,\gamma,q)_{m_1} & \mathscr{M}(d,w,\alpha,\beta,\gamma,q)_{n_1+k_2} \\ \mathscr{M}(d,w,\alpha,\beta,\gamma,q)_{m_1-k_2} & \mathscr{M}(d,w,\alpha,\beta,\gamma,q)_{n_1}\end{bmatrix} \in \mathbb{Q}(\mathfrak{X}).
\end{aligned}$$

Let $\Delta(d,w,\alpha,\gamma,m,n,k,\rho,q) = \Delta(d,w,\alpha,\iota^d(1),\gamma,m,n,k,\rho,q)$ for a reduced multimonomial index (d,w,α,γ).

Then, we obtain the following strict inequality by monomial convolutions and weighted q-multinomial coefficients. This appears as Theorem 1.16 in Section 1 with a different notation.

Theorem 18.15. *Suppose a multimonomial index* $\mu = (d,w,\alpha,\gamma)$. *Let* $\delta_l = \mathrm{lcm}(\alpha)$, $s = (0,\infty)$, $l = 1$, $\succ = >_{\alpha,\gamma,q}$, *and* $\rho \in \mathbb{Z}^l_{\geq 1}$. *Then, for each fitting* (s,l,m,n,k), *we have the following strict inequality:*

$$\Delta(d,w,\alpha,\gamma,m,n,k,\rho,q) \succ 0.$$

Proof. The assertion holds, because Claim 2 of Proposition 18.13 gives the $\succ$-merged-log-concave parcel $\mathscr{F} = \Lambda(s,l,w,\succ,f_s,\rho,\left(q^{\delta_l}\right),\mathfrak{U}_{\alpha,\gamma,q})$ such that $\mathscr{M}(d,w,\alpha,\gamma,q,v) = \mathscr{Z}_{\mathscr{F}}(v)$. □

Example 18.16. Consider the multimonomial index (d,w,α,γ) such that $d=2$, $w=(1)$, $\alpha=(2,3)$, and $\gamma = ((0,0,0),(0,1,0))$. Then, $>_{\alpha,\gamma,q} = >_q$. Let $x = \left(q^{\mathrm{lcm}(\alpha)}\right)$. Then, we have

$$(m)_x^w \mathscr{M}(d,w,\alpha,\gamma,q)_m >_q 0$$

for each $m \in \mathbb{Z}_{\geq 0}$. For example,

$$\begin{aligned}
(0)_x\mathscr{M}(d,w,\alpha,\gamma,q)_0 &= 1,\\
(1)_x\mathscr{M}(d,w,\alpha,\gamma,q)_1 &= q^6+q^4+q^3+q^2+1,\\
(2)_x\mathscr{M}(d,w,\alpha,\gamma,q)_2 &= q^{16}+q^{15}+q^{14}+q^{13}+3q^{12}+q^{11}+2q^{10}+2q^9+3q^8+q^7+3q^6+q^5+2q^4+q^3+q^2\\
&\quad +1,\\
(3)_x\mathscr{M}(d,w,\alpha,\gamma,q)_3 &= q^{31}+q^{30}+q^{29}+2q^{28}+3q^{27}+3q^{26}+3q^{25}+5q^{24}+4q^{23}+6q^{22}\\
&\quad +5q^{21}+7q^{20}+5q^{19}+9q^{18}+5q^{17}+8q^{16}+6q^{15}+8q^{14}+4q^{13}+8q^{12}\\
&\quad +4q^{11}+6q^{10}+4q^9+5q^8+2q^7+4q^6+q^5+2q^4+q^3+q^2+1.
\end{aligned}$$

Furthermore, let $k=(0,1)$ and $\rho=(1)$. Then, Theorem 18.15 gives the following q-polynomials with positive coefficients:

$$\begin{aligned}
\Delta(d,w,\alpha,\gamma,(0),(0),k,\rho,q) &= 1;\\
\Delta(d,w,\alpha,\gamma,(1),(1),k,\rho,q) &= q^{18}+q^{16}+q^{15}+2q^{14}+q^{13}+3q^{12}+q^{11}+3q^{10}\\
&\quad +2q^9+2q^8+q^7+3q^6+q^5+q^4+q^3+q^2;\\
\Delta(d,w,\alpha,\gamma,(2),(2),k,\rho,q) &= q^{44}+q^{43}+2q^{42}+2q^{41}+5q^{40}+4q^{39}+7q^{38}+7q^{37}+13q^{36}\\
&\quad +10q^{35}+17q^{34}+16q^{33}+22q^{32}+19q^{31}+28q^{30}+22q^{29}+31q^{28}\\
&\quad +28q^{27}+32q^{26}+27q^{25}+37q^{24}+27q^{23}+32q^{22}+28q^{21}+31q^{20}\\
&\quad +22q^{19}+28q^{18}+19q^{17}+22q^{16}+16q^{15}+17q^{14}+10q^{13}\\
&\quad +13q^{12}+7q^{11}+7q^{10}+4q^9+5q^8+2q^7+2q^6+q^5+q^4;\\
\Delta(d,w,\alpha,\gamma,(3),(3),k,\rho,q) &= q^{80}+q^{79}+2q^{78}+3q^{77}+6q^{76}+6q^{75}+12q^{74}+14q^{73}+21q^{72}+25q^{71}\\
&\quad +37q^{70}+40q^{69}+56q^{68}+64q^{67}+83q^{66}+92q^{65}+118q^{64}+129q^{63}\\
&\quad +158q^{62}+171q^{61}+206q^{60}+216q^{59}+259q^{58}+271q^{57}+310q^{56}\\
&\quad +320q^{55}+373q^{54}+367q^{53}+418q^{52}+420q^{51}+464q^{50}+452q^{49}+508q^{48}\\
&\quad +478q^{47}+529q^{46}+504q^{45}+539q^{44}+499q^{43}+550q^{42}+490q^{41}+525q^{40}\\
&\quad +475q^{39}+502q^{38}+435q^{37}+467q^{36}+396q^{35}+416q^{34}+352q^{33}+364q^{32}\\
&\quad +296q^{31}+313q^{30}+248q^{29}+252q^{28}+200q^{27}+205q^{26}+152q^{25}+156q^{24}\\
&\quad +115q^{23}+115q^{22}+82q^{21}+81q^{20}+55q^{19}+56q^{18}+36q^{17}+34q^{16}\\
&\quad +22q^{15}+21q^{14}+12q^{13}+11q^{12}+6q^{11}+6q^{10}+3q^9+2q^8+q^7+q^6.
\end{aligned}$$

In Example 18.16, $\Delta(d,w,\alpha,\gamma,(i),(i),k,\rho,q)$ are not log-concave q-polynomials for $i \in [\![3]\!]$. Even

$$\Delta(d,w,\iota^2(1),\gamma,(1),(1),k,\rho,q) = q^3+q^2+2q$$

is not a log-concave q-polynomial either. However, we state the following conjecture.

Conjecture 18.17. *Let $\lambda_1,\lambda_2 \in \mathbb{Z}_{\geq 1}$ and $\lambda_3 \in \mathbb{Z}_{\geq 0}$. Consider the multimonomial index $\left(d,w,\alpha,\beta_{\lambda_1},\gamma\right)$ such that $d=2$, $w=(1)$, $\alpha=\iota^d(1)$, $\beta_{\lambda_1}=\iota^d(\lambda_1)$, and $\gamma=\left(\left(\frac{1}{2},-\frac{1}{2},0\right),(0,1,0)\right)$. Then, the merged determinant*

$$\Delta(d,w,\alpha,\beta_{\lambda_1},\gamma,(\lambda_3),(\lambda_3),(0,1),(\lambda_2),q)$$

is a log-concave q-polynomial.

For the multimonomial index $\left(d,w,\alpha,\beta_{\lambda_1},\gamma\right)$ in Conjecture 18.17, suppose $\beta'_{\lambda_1}=(\lambda_1,-\lambda_1)$ and $\kappa=(-1,2)$. By Proposition 18.7, we have the (q,t)-analog $M(d,w,\alpha,\beta_{\lambda_1},\gamma(\beta'_{\lambda_1},\kappa),q,z)$ of the eta product

$$1=\eta(\tau)^{\lambda_1}\cdot\eta(\tau)^{-\lambda_1}=E_{d,\alpha,\beta'_{\lambda_1}}(\tau).$$

We now obtain the merged determinant in Conjecture 18.17 by the monomial convolution $\mathscr{M}(d,w,\alpha,\beta_{\lambda_1},\gamma(\beta'_{\lambda_1},\kappa),q,v)$ of the (q,t)-analog (see Section 1.10).

18.5 Examples and conjectures for merged determinants of monomial convolutions

We consider multimonomial indices (d,w,α,γ) such that $\gamma=\iota^d\left(\left(\frac{1}{2},-\frac{1}{2},0\right)\right)$, $\iota^d((0,0,0))$, or $\iota^d((0,1,0))$. They give the eta products $\prod_{i\in[\![d]\!]}\eta(\alpha_i\tau)$ or $\prod_{i\in[\![d]\!]}\eta(\alpha_i\tau)^{-1}$ in Proposition 18.7, multiplied by the constant overall factors.

Example 18.18. For $d\in\mathbb{Z}_{\geq 1}$ and $\alpha\in\mathbb{Z}^d_{\geq 1}$, $\gamma_1=\iota^d((0,0,0))$ and $\gamma_2=\iota^d((0,1,0))$ give the same eta products $\prod_{i\in[\![d]\!]}\eta(\alpha_i\tau)^{-1}$. But, their merged determinants differ non-trivially.

Let $s=(0,\infty)$, $l=1$, $d=2$, $w=(1)$, $\alpha=(1,2)$, $k=(0,1)$, and $\rho=(1)$. Also, let $m_i=(i)\in\mathbb{Z}^l_{\geq 0}$ so that (s,l,m_i,m_i,k) is fitting for each $i\in\mathbb{Z}_{\geq 0}$. First, γ_1 gives

$$\begin{aligned}
\Delta(d,w,\alpha,\gamma_1,m_1,m_1,k,\rho,q) &= 1+2q+3q^2+2q^3+q^4,\\
\Delta(d,w,\alpha,\gamma_1,m_2,m_2,k,\rho,q) &= 1+2q+6q^2+8q^3+13q^4+14q^5+11q^6+12q^7+7q^8+4q^9+3q^{10},\\
\Delta(d,w,\alpha,\gamma_1,m_3,m_3,k,\rho,q) &= 1+2q+6q^2+12q^3+21q^4+32q^5+49q^6+58q^7+69q^8+78q^9+77q^{10}\\
&\quad +76q^{11}+68q^{12}+58q^{13}+44q^{14}+34q^{15}+22q^{16}+12q^{17}+7q^{18}+2q^{19}+q^{20}.
\end{aligned}$$

Second, γ_2 gives

$$\begin{aligned}
\Delta(d,w,\alpha,\gamma_2,m_1,m_1,k,\rho,q) &= 2q^3+2q^4+2q^5+3q^6,\\
\Delta(d,w,\alpha,\gamma_2,m_2,m_2,k,\rho,q) &= 3q^6+4q^7+7q^8+12q^9+11q^{10}+14q^{11}+13q^{12}+8q^{13}+6q^{14}+2q^{15}+q^{16},\\
\Delta(d,w,\alpha,\gamma_2,m_3,m_3,k,\rho,q) &= 4q^9+6q^{10}+12q^{11}+23q^{12}+30q^{13}+46q^{14}+58q^{15}+69q^{16}+76q^{17}+80q^{18}\\
&\quad +78q^{19}+67q^{20}+60q^{21}+44q^{22}+32q^{23}+22q^{24}+12q^{25}+7q^{26}+2q^{27}+q^{28}.
\end{aligned}$$

We define the following differences by merged determinants of monomial convolutions.

Definition 18.19. *Suppose a multimonomial index* $(d,w,\alpha,\beta,\gamma)$*. Let* $k=(0,1)$*,* $\delta=\mathrm{lcm}(\alpha)$*, and* $\lambda_1,\lambda_2,\lambda_3\in\mathbb{Z}_{\geq 0}$*. Then, in* $\mathbb{Q}[q^{\pm u_{\alpha,\gamma}}]$*, we define*

$$\begin{aligned}\xi(d,w,\alpha,\beta,\gamma,\lambda_1,\lambda_2,\lambda_3,\rho,q) &= \Delta(d,w,\alpha,\beta,\gamma,(\lambda_1+\lambda_3),(\lambda_1+\lambda_2+\lambda_3),k,\rho,q)\\ &\quad -\Delta(d,w,\alpha,\beta,\gamma,(\lambda_1),(\lambda_1+\lambda_2),k,\rho,q).\end{aligned}$$

If $\beta=\iota^d(1)$*, then let* $\xi(d,w,\alpha,\gamma,\lambda_1,\lambda_2,\lambda_3,\rho,q)=\xi(d,w,\alpha,\beta,\gamma,\lambda_1,\lambda_2,\lambda_3,\rho,q)$*.*

We then conjecture the following positivity of ξ.

Conjecture 18.20. *Let* $d\in\mathbb{Z}_{\geq 2}$*,* $w,\rho\in\mathbb{Z}^1_{\geq 1}$*, and* $\alpha\in\mathbb{Z}^d_{\geq 1}$*. Consider the multimonomial index* (d,w,α,γ) *such that* $\gamma=\iota^d((0,0,0))$*. Then, for each* $\lambda_1,\lambda_2\in\mathbb{Z}_{\geq 0}$*, we have*

$$\xi(d,w,\alpha,\gamma,\lambda_1,\lambda_2,1,\rho,q)>_q 0.$$

Example 18.21. Suppose the multimonomial index (d,w,α,γ) such that $d=2$, $w=(1)$, $\alpha=(1,1)$, and $\gamma=\iota^d((0,0,0))$. Let $k=(0,1)$ and $\rho=(1)$. Then,

$$\begin{aligned}
\Delta(d,w,\alpha,\gamma,(0),(0),k,\rho,q) &= 1,\\
\Delta(d,w,\alpha,\gamma,(1),(1),k,\rho,q) &= 3q+1,\\
\Delta(d,w,\alpha,\gamma,(2),(2),k,\rho,q) &= q^4+3q^3+8q^2+3q+1,\\
\Delta(d,w,\alpha,\gamma,(3),(3),k,\rho,q) &= 3q^7+5q^6+12q^5+14q^4+18q^3+8q^2+3q+1,\\
\Delta(d,w,\alpha,\gamma,(4),(4),k,\rho,q) &= q^{12}+3q^{11}+12q^{10}+18q^9+30q^8+39q^7\\
&\quad +46q^6+39q^5+38q^4+18q^3+8q^2+3q+1.
\end{aligned}$$

Thus, the following are q-polynomials with positive integer coefficients:

$$\begin{aligned}
&\xi(d,w,\alpha,\gamma,0,0,1,\rho,q)=3q;\\
&\xi(d,w,\alpha,\gamma,1,0,1,\rho,q)=q^4+3q^3+8q^2;\\
&\xi(d,w,\alpha,\gamma,2,0,1,\rho,q)=3q^7+5q^6+12q^5+13q^4+15q^3;\\
&\xi(d,w,\alpha,\gamma,3,0,1,\rho,q)=q^{12}+3q^{11}+12q^{10}+18q^9+30q^8+36q^7+41q^6+27q^5+24q^4.
\end{aligned}$$

Example 18.22. Conjecture 18.20 does not extend to the case of $d=1$. Suppose the multimonomial index (d,w,α,γ) such that $d=1$, $w=(1)$, $\alpha=(2)$, and $\gamma=((0,0,0))$. Let $k=(0,1)$ and $\rho=(1)$. Then, we have $\xi(d,w,\alpha,\gamma,2,1,1,\rho,q)\not>_q 0$, since

$$\begin{aligned}
\Delta(d,w,\alpha,\gamma,(2),(3),k,\rho,q) &= q^6+q^4,\\
\Delta(d,w,\alpha,\gamma,(3),(4),k,\rho,q) &= q^8+q^6.
\end{aligned}$$

Example 18.23. Conjecture 18.20 does not extend to the case of $\gamma=\iota^d((0,1,0))$ either. Consider the multimonomial index (d,w,α,γ) with $d=2$, $w=(1)$, $\alpha=(2,1)$, and $\gamma=\iota^d((0,1,0))$. Let $k=(0,1)$ and $\rho=(1)$. Then, $\xi(d,w,\alpha,\gamma,0,0,1,\rho,q)\not>_q 0$, because

$$\begin{aligned}
\Delta(d,w,\alpha,\gamma,(0),(0),k,\rho,q) &= 1,\\
\Delta(d,w,\alpha,\gamma,(1),(1),k,\rho,q) &= 3q^6+2q^5+2q^4+2q^3.
\end{aligned}$$

In Claim 2 of Theorem 18.3, $\Delta(d,w,\alpha,\beta,\gamma,m,n,k,\rho,q)\in\mathbb{Q}[q^{\pm u_{\alpha,\gamma}}]$. By $C_{q^{u_{\alpha,\gamma}}}$ in Definition 14.4, we define the following notation to state another conjecture for $\gamma=\iota^d((0,1,0))$ of $d\in\mathbb{Z}_{\geq 2}$.

Definition 18.24. *Consider a multimonomial index* $(d,w,\alpha,\beta,\gamma)$*. Let* $l=1$*,* $m,n\in\mathbb{Z}^l$*,* $k=(0,1)$*, and* $\lambda_1,\lambda_2,\lambda_3\in\mathbb{Z}_{\geq 0}$*. Then, in* $\mathbb{Q}[q^{u_{\alpha,\gamma}}]$*, we define*

$$\begin{aligned}\Delta_C(d,w,\alpha,\beta,\gamma,m,n,k,\rho,q) &= C_{q^{u_{\alpha,\gamma}}}(\Delta(d,w,\alpha,\beta,\gamma,m,n,k,\rho,q)),\\ \xi_C(d,w,\alpha,\beta,\gamma,\lambda_1,\lambda_2,\lambda_3,\rho,q) &= \Delta_C(d,w,\alpha,\beta,\gamma,(\lambda_1+\lambda_3),(\lambda_1+\lambda_2+\lambda_3),k,\rho,q)\\ &\quad -\Delta_C(d,w,\alpha,\beta,\gamma,(\lambda_1),(\lambda_1+\lambda_2),k,\rho,q).\end{aligned}$$

If $\beta=\iota^d(1)$*, then let* $\xi_C(d,w,\alpha,\gamma,\lambda_1,\lambda_2,\lambda_3,\rho,q)=\xi_C(d,w,\alpha,\beta,\gamma,\lambda_1,\lambda_2,\lambda_3,\rho,q)$*.*

Let us recall the following *generalized Narayana numbers* [Guy].

Definition 18.25. *Let* $d\in\mathbb{Z}_{\geq 1}$ *and* $h,\lambda\in\mathbb{Z}_{\geq 0}$*. Then, the generalized Narayana number* $N(\lambda,d,h)$ *satisfies*

$$N(\lambda,d,h)=\frac{\lambda+1}{d}\binom{d}{h}\binom{d}{h-1-\lambda}.$$

For instance, $N(0,d,h)$ are Narayana numbers, which refine Catalan numbers. We also employ the following numbers.

Definition 18.26. *Let* $d\in\mathbb{Z}_{\geq 2}$*,* $h_1,h_2\in\mathbb{Z}$*, and* $\alpha\in\mathbb{Z}^d_{\geq 1}$*. Then, we define*

$$p(d,h_1,h_2,\alpha)=(d-\mathrm{L}_{h_2}(\alpha))h_2+h_1\mathrm{L}_{h_2}(\alpha)\in\mathbb{Z}.$$

Notice that $h_2>h_1\geq 1$ implies $p(d,h_1,h_2,\alpha)\geq 2$. We now conjecture the following positivity on ξ_C, periodicity on Δ_C and $p(d,h_1,h_2,\alpha)$, and equality on Δ_C and $N(\lambda,d,h)$.

In particular, Conjectures 18.20 and 18.27 claim different positivities for $\iota^d((0,0,0))$ and $\iota^d((0,1,0))$, which give the same eta products (see Example 18.18).

Conjecture 18.27. *Let* $d\in\mathbb{Z}_{\geq 2}$*,* $w,\rho\in\mathbb{Z}^1_{\geq 1}$*,* $\alpha\in\mathbb{Z}^d_{\geq 1}$*, and* $k=(0,1)$*. For* $i\in[\![2]\!]$*, consider multimonomial indices* (d,w,α,γ_i) *such that* $\gamma_1=\iota^d((0,1,0))$ *and* $\gamma_2=\iota^d((\frac{1}{2},-\frac{1}{2},0))$*.*

1. *If* $\lambda_1,\lambda_2\in\mathbb{Z}_{\geq 0}$*, then we have*

$$\xi_C(d,w,\alpha,\gamma_1,\lambda_1,\lambda_2,1,\rho,q)>_q 0.$$

2. *Let* $h_1,h_2\in\mathbb{Z}_{\geq 1}$ *with* $h_2>h_1$*. Suppose* $\alpha_i=h_1$ *or* $\alpha_i=h_2$ *for each* $i\in[\![d]\!]$*. Then, whenever* $\lambda_1,\lambda_2\in\mathbb{Z}_{\geq 0}$*, we have the following periodicity of* Δ_C*: first, if* $\mathrm{L}_{h_1}(\alpha)>0$ *and* $\mathrm{L}_{h_2}(\alpha)>0$*, then*

$$\xi_C(d,w,\alpha,\gamma_2,\lambda_1,\lambda_2,p(d,h_1,h_2,\alpha),\rho,0)=0;$$

second, if $\mathrm{L}_{h_1}(\alpha)=d$ *or* $\mathrm{L}_{h_2}(\alpha)=d$*, then*

$$\xi_C(d,w,\alpha,\gamma_2,\lambda_1,\lambda_2,d,\rho,0)=0.$$

3. *Assume* $\alpha=\iota^d(1)$ *and* $\lambda,h\in\mathbb{Z}_{\geq 0}$ *such that* $h\leq d-\lambda$*. Then, we have*

$$\Delta_C(d,w,\alpha,\gamma_2,(h),(h+\lambda),k,\rho,0)=N(\lambda,d+1,h+1+\lambda).$$

In particular, weighted q-multinomial coefficients would extend the generalized Narayana numbers $N(\lambda, d+1, h+1+\lambda)$ by $\Delta_C(d,w,\alpha,\gamma_2,(h),(h+\lambda),k,\rho,0) > 0$.

Example 18.21 supports Claim 1 in Conjecture 18.27. To see this, suppose multimonomial indices (d,w,α,γ) of $w=(1)$, $d=2$, $\alpha=\iota^d(1)$, and $\gamma=\iota^d((0,1,0))$. Also, let $k=(0,1)$, $\rho=(1)$, and $\gamma'=\iota^d((0,0,0))$. Then, we have $\Delta_C(d,w,\alpha,\gamma,(i),(i),k,\rho,q)=\Delta_C(d,w,\alpha,\gamma',(i),(i),k,\rho,q)$, since α is flat. Example 18.23 is also consistent with Claim 1 in Conjecture 18.27, as $\Delta_C(d,w,\alpha,\gamma,(1),(1),k,\rho,q)=3q^3+2q^2+2q+2$.

The following supports Claims 2 and 3 in Conjecture 18.27.

Example 18.28. Consider the multimonomial index (d,w,α,γ) such that $d=3$, $w=(1)$, $\alpha=\iota^d(1)$, and $\gamma=\iota^d((\frac{1}{2},-\frac{1}{2},0))$. Let $k=(0,1)$ and $\rho=(1)$. Then,

$$\begin{aligned}
\Delta(d,w,\alpha,\gamma,(0),(0),k,\rho,q) &= 1,\\
\Delta(d,w,\alpha,\gamma,(1),(1),k,\rho,q) &= 3q+6,\\
\Delta(d,w,\alpha,\gamma,(2),(2),k,\rho,q) &= 6q^4+18q^3+33q^2+18q+6,\\
\Delta(d,w,\alpha,\gamma,(3),(3),k,\rho,q) &= 10q^9+35q^8+88q^7+155q^6+162q^5+144q^4+83q^3+43q^2+8q+1,\\
\Delta(d,w,\alpha,\gamma,(4),(4),k,\rho,q) &= 15q^{16}+57q^{15}+162q^{14}+357q^{13}+642q^{12}+858q^{11}+1041q^{10}\\
&\quad +1041q^9+912q^8+678q^7+447q^6+222q^5+96q^4+27q^3+6q^2,\\
\Delta(d,w,\alpha,\gamma,(5),(5),k,\rho,q) &= 21q^{25}+84q^{24}+255q^{23}+618q^{22}+1296q^{21}+2328q^{20}+3528q^{19}\\
&\quad +4905q^{18}+6105q^{17}+6951q^{16}+7161q^{15}+6882q^{14}+5958q^{13}+4791q^{12}\\
&\quad +3450q^{11}+2280q^{10}+1308q^9+690q^8+294q^7+111q^6+27q^5+6q^4,\\
\Delta(d,w,\alpha,\gamma,(6),(6),k,\rho,q) &= 28q^{36}+116q^{35}+367q^{34}+938q^{33}+2114q^{32}+4229q^{31}+7638q^{30}\\
&\quad +12248q^{29}+18374q^{28}+25457q^{27}+33106q^{26}+40206q^{25}\\
&\quad +46214q^{24}+49983q^{23}+51342q^{22}+49690q^{21}+45663q^{20}\\
&\quad +39534q^{19}+32462q^{18}+24969q^{17}+18121q^{16}+12227q^{15}+7745q^{14}\\
&\quad +4463q^{13}+2382q^{12}+1124q^{11}+476q^{10}+164q^9+52q^8+8q^7+q^6.
\end{aligned}$$

We have $\mathrm{L}_1(\alpha)=d$. Thus, Claim 2 and 3 in Conjecture 18.27 agree with $\xi_C(d,w,\alpha,\gamma,i,0,3,\rho,0)=0$ for $i\in[\![0,3]\!]$, and with

$$\begin{aligned}
\Delta_C(d,w,\alpha,\gamma,(0),(0),k,\rho,0) &= 1 = N(0,4,1),\\
\Delta_C(d,w,\alpha,\gamma,(1),(1),k,\rho,0) &= 6 = N(0,4,2),\\
\Delta_C(d,w,\alpha,\gamma,(2),(2),k,\rho,0) &= 6 = N(0,4,3),\\
\Delta_C(d,w,\alpha,\gamma,(3),(3),k,\rho,0) &= 1 = N(0,4,4).
\end{aligned}$$

Example 18.29. Suppose the multimonomial index (d,w,α,γ) such that $d=4$, $w=(1)$, $\alpha=\iota^d(1)$, and $\gamma=\iota^d\left(\left(\frac{1}{2},-\frac{1}{2},0\right)\right)$. Let $\rho=(1)$, $k=(0,1)$, and $\lambda=2$. Then, as in Claim 2 of Conjecture 18.27, we have the following periodicity:

$$\begin{aligned}
\Delta_C(d,w,\alpha,\gamma,(0),(0+\lambda),k,\rho,0) &= \Delta_C(d,w,\alpha,\gamma,(4),(4+\lambda),k,\rho,0)=6;\\
\Delta_C(d,w,\alpha,\gamma,(1),(1+\lambda),k,\rho,0) &= \Delta_C(d,w,\alpha,\gamma,(5),(5+\lambda),k,\rho,0)=15;\\
\Delta_C(d,w,\alpha,\gamma,(2),(2+\lambda),k,\rho,0) &= \Delta_C(d,w,\alpha,\gamma,(6),(6+\lambda),k,\rho,0)=6;\\
\Delta_C(d,w,\alpha,\gamma,(3),(3+\lambda),k,\rho,0) &= \Delta_C(d,w,\alpha,\gamma,(7),(7+\lambda),k,\rho,0)=16.
\end{aligned}$$

In particular, the generalized Narayana numbers in Claim 3 of Conjecture 18.27 give the first three numbers in the above, since $0 \leq h \leq d - \lambda = 2$ and

$$\begin{aligned}N(\lambda, d+1, 0+1+\lambda) &= N(2,5,3) = 6,\\ N(\lambda, d+1, 1+1+\lambda) &= N(2,5,4) = 15,\\ N(\lambda, d+1, 2+1+\lambda) &= N(2,5,5) = 6.\end{aligned}$$

For another example, let $\lambda = 3$. Then, we obtain the following periodicity:

$$\begin{aligned}\Delta_C(d, w, \alpha, \gamma, (0), (0+\lambda), k, \rho, 0) &= \Delta_C(d, w, \alpha, \gamma, (4), (4+\lambda), k, \rho, 0) = 4;\\ \Delta_C(d, w, \alpha, \gamma, (1), (1+\lambda), k, \rho, 0) &= \Delta_C(d, w, \alpha, \gamma, (5), (5+\lambda), k, \rho, 0) = 4;\\ \Delta_C(d, w, \alpha, \gamma, (2), (2+\lambda), k, \rho, 0) &= \Delta_C(d, w, \alpha, \gamma, (6), (6+\lambda), k, \rho, 0) = 24;\\ \Delta_C(d, w, \alpha, \gamma, (3), (3+\lambda), k, \rho, 0) &= \Delta_C(d, w, \alpha, \gamma, (7), (7+\lambda), k, \rho, 0) = 24.\end{aligned}$$

Then, the generalized Narayana numbers in Claim 3 of Conjecture 18.27 account the first two numbers, because $0 \leq h \leq d - \lambda = 1$ and

$$\begin{aligned}N(\lambda, d+1, 0+1+\lambda) &= N(3,5,4) = 4,\\ N(\lambda, d+1, 1+1+\lambda) &= N(3,5,5) = 4.\end{aligned}$$

We compute the following for some non-flat α in Claim 2 of Conjecture 18.27.

Example 18.30. Consider the multimonomial index (d, w, α, γ) such that $d = 2$, $w = (1)$, $\alpha = (1,2)$, and $\gamma = \iota^d((\frac{1}{2}, -\frac{1}{2}, 0))$. Let $\rho = (1)$ and $k = (0,1)$. Also, let $h_1 = 1$ and $h_2 = 2$. Then, $\mathrm{L}_2(\alpha) = 1$ gives $p(d, h_1, h_2, \alpha) = 2 \cdot 2 - 1 \cdot 1 = 3$. Furthermore, Claim 2 of Conjecture 18.27 is consistent with

$$\begin{aligned}\Delta_C(d, w, \alpha, \gamma, (0), (0), k, \rho, 0) &= \Delta_C(d, w, \alpha, \gamma, (3), (3), k, \rho, 0) = 1,\\ \Delta_C(d, w, \alpha, \gamma, (1), (1), k, \rho, 0) &= \Delta_C(d, w, \alpha, \gamma, (4), (4), k, \rho, 0) = 3,\\ \Delta_C(d, w, \alpha, \gamma, (2), (2), k, \rho, 0) &= \Delta_C(d, w, \alpha, \gamma, (5), (5), k, \rho, 0) = 1.\end{aligned}$$

19 Monomial convolutions and ideal boson–fermion gases

Generalizing Section 1.11, Section 19 considers some ideal (mixed) boson–fermion gases with or without Casimir energies by monomial convolutions. Thus, we obtain statistical-mechanical phase transitions by the merged-log-concavity.

Unless stated otherwise, Section 19 assumes the following. Let $s = (0, \infty)$, $l = 1$, $w = (1)$, and $v \in \mathbb{Q}$. Also, as in Section 1.11, let $q = e^{-\beta}$ by the thermodynamic beta $\beta > 0$ and $t = e^{-\mu'}$ by $\mu' = -\mu\beta$ of the chemical potential $\mu < 0$.

19.1 Monomial convolutions without Casimir energies

We have the following systems of boson–fermion gases without Casimir energies by monomial convolutions. Suppose integers $0 \leq d_1 \leq d_2$ such that $d_2 \in \mathbb{Z}_{\geq 1}$ and $\kappa \in \mathbb{Q}^{d_2}$. Consider the boson–fermion system $M_{d_1,d_2}(\kappa)$ that has sub-systems $B(1, \kappa_\lambda)$ for $\lambda \in [\![d_1]\!]$ and sub-systems $F(1, \kappa_\lambda)$ for $\lambda \in [\![d_1+1, d_2]\!]$ with negligible interactions among sub-systems. Thus, $M_{d_1,d_2}(\kappa)$ has the grand canonical partition function

$$\mathscr{Z}_{M_{d_1,d_2}(\kappa)}(q,t) = \prod_{\lambda \in [\![d_1]\!]} \mathrm{Tr}\left(e^{-\beta H_{b,\kappa_\lambda}} \cdot e^{-\mu' N_b}\right) \cdot \prod_{\lambda \in [\![d_1+1, d_2]\!]} \mathrm{Tr}\left(e^{-\beta H_{f,\kappa_\lambda}} \cdot e^{-\mu' N_f}\right).$$

Moreover, $\alpha = \iota^{d_2}(1)$ gives the multimonomial index $(d_2, w, \alpha, \gamma_\kappa)$ such that

$$\gamma_{\kappa,\lambda} = \begin{cases} \gamma_{b,\kappa_\lambda} & \text{for each } \lambda \in [\![d_1]\!], \\ \gamma_{f,\kappa_\lambda} & \text{for each } \lambda \in [\![d_1+1, d_2]\!]. \end{cases}$$

Then, by equations (1.11.4) and (1.11.5), we realize

$$\mathscr{Z}_{M_{d_1,d_2}(\kappa)}(q,t) = \mathscr{M}(d_2, w, \alpha, \gamma_\kappa, q, t).$$

19.2 Monomial convolutions with Casimir energies

Sections 1.11 and 19.1 ignore *zero-point energies*, a common practice in statistical mechanics [KapGal]. However, let us not ignore them. By monomial convolutions, we incorporate the Casimir energies as the following Ramanujan summation of the zero-point energy sums of $B(1,v)$ and $F(1,v)$.

Suppose an entire function $a(y)$ of $y \in \mathbb{C}$ such that the exponential type of $a(y)$ is less than π. Then, by the Bernoulli numbers $B_{\lambda+1}$, we recall the Ramanujan summation [Can]:

$$\sum_{\lambda\in\mathbb{Z}_{\geq 1}}^{\mathscr{R}} a(\lambda) = \int_0^1 a(y)dy - \frac{1}{2}a(0) - \sum_{\lambda\in\mathbb{Z}_{\geq 1}} a^{(\lambda)}(0)\frac{B_{\lambda+1}}{(\lambda+1)!}. \tag{19.2.1}$$

For example, $a_v(y) = y - v$ gives $\sum^{\mathscr{R}}_{\lambda\in\mathbb{Z}_{\geq 1}} a_v(\lambda) = \frac{5}{12} - \frac{v}{2}$, which is $-\frac{1}{12}$ [Ram] when $v = 1$.

19.2.1 Ideal boson gases

We have the following operators with zero-point energies.

Definition 19.1. *For $v \in \mathbb{Q}$, let $\mathscr{H}_{b,v}$ denote the Hamiltonian operator such that*

$$\begin{aligned} \mathscr{H}_{b,v} &= \sum_{\lambda\in\mathbb{Z}_{\geq 1}} \frac{\varepsilon_{v,\lambda}}{2}\{a^\dagger_{b,\lambda}, a_{b,\lambda}\} \\ &= \sum_{\lambda\in\mathbb{Z}_{\geq 1}} \left(\varepsilon_{v,\lambda} a^\dagger_{b,\lambda} a_{b,\lambda} + \frac{\varepsilon_{v,\lambda}}{2}\right). \end{aligned} \tag{19.2.2}$$

If $u \in \mathbb{Z}_{\geq 1}$, then let $\mathscr{H}_{b,v,u}$ and $N_{b,u}$ denote the Hamiltonian and number operators such that

$$\begin{aligned} \mathscr{H}_{b,v,u} &= \sum_{\lambda\in[\![u]\!]} \left(\varepsilon_{v,\lambda} a^\dagger_{b,\lambda} a_{b,\lambda} + \frac{\varepsilon_{v,\lambda}}{2}\right), \\ N_{b,u} &= \sum_{\lambda\in[\![u]\!]} a^\dagger_{b,\lambda} a_{b,\lambda}. \end{aligned}$$

Also, let $\gamma^{\mathscr{R}}_{b,v} = \left(\left(0, 1-v, \frac{5}{24} - \frac{v}{4}\right)\right)$.

Consider the boson system $\mathscr{B}(1,v)$ of $\mathscr{H}_{b,v}$ and N_b in Definition 1.25. Unlike in equation (1.11.3), $\frac{\varepsilon_{v,\lambda}}{2}$ in equation (19.2.2) represent zero-point energies. Thus, $\mathscr{B}(1,v)$ extends $B(1,v)$ by the zero-point energies. Also, $\mathscr{B}(1,v)$ proposes the grand canonical partition function

$$\mathscr{Z}_{\mathscr{B}(1,v)}(q,t) = \operatorname{Tr}\left(e^{-\beta\mathscr{H}_{b,v}} \cdot e^{-\mu' N_b}\right).$$

However, $\mathscr{Z}_{\mathscr{B}(1,v)}(q,t)$ has $q^{\frac{\sum_{1\leq\lambda}\varepsilon_{v,\lambda}}{2}}$ of the divergent zero-point energy sum $\frac{\sum_{1\leq\lambda}\varepsilon_{v,\lambda}}{2}$. This would make $\mathscr{Z}_{\mathscr{B}(1,v)}(q,t)=0$ by $0<q<1$. Thus, we consider the regularized grand canonical partition function $\mathscr{Z}^{\mathscr{R}}_{\mathscr{B}(1,v)}(q,t)$ of $\mathscr{B}(1,v)$ by the Ramanujan summation and a monomial convolution as follows.

For each $u\in\mathbb{Z}_{\geq 1}$, there exists the boson system $\mathscr{B}(1,v,u)$ of $\mathscr{H}_{b,v,u}$ and $N_{b,u}$. Also, for each $\lambda\in\mathbb{Z}_{\geq 1}$ and $n_\lambda\in\mathbb{Z}_{\geq 0}$, we have $\langle n_\lambda|e^{-\beta\varepsilon_{v,\lambda}\left(a^\dagger_\lambda\alpha_\lambda+\frac{1}{2}\right)}\cdot e^{-\mu' a^\dagger_\lambda\alpha_\lambda}|n_\lambda\rangle = e^{-\beta\varepsilon_{v,\lambda}n_\lambda}\cdot e^{-\mu' n_\lambda}\cdot e^{-\beta\frac{\varepsilon_{v,\lambda}}{2}}$. Thus, by $\prod_{\lambda\in[\![u]\!]}e^{-\beta\frac{\varepsilon_{v,\lambda}}{2}}=q^{\frac{\sum_{\lambda\in[\![u]\!]}\varepsilon_{v,\lambda}}{2}}$, $\mathscr{B}(1,v,u)$ has the grand canonical partition function

$$\mathscr{Z}_{\mathscr{B}(1,v,u)}(q,t)=\operatorname{Tr}\left(e^{-\beta\mathscr{H}_{b,v,u}}\cdot e^{-\mu' N_{b,u}}\right)=(q^{1-v}t;q)_u^{-1}\cdot q^{\sum_{\lambda\in[\![u]\!]}\frac{\varepsilon_{v,\lambda}}{2}}. \tag{19.2.3}$$

In equations (19.2.3), $u\to\infty$ yields the divergent zero-point energy sum

$$\lim_{u\to\infty}\frac{1}{2}\sum_{\lambda\in[\![u]\!]}\varepsilon_{v,\lambda}.$$

But, the Ramanujan summation (19.2.1) gives the following regularization (c.f. [Pol, Section 1.3] for $v=1$):

$$\lim_{u\to\infty}\frac{1}{2}\sum_{\lambda\in[\![u]\!]}\varepsilon_{v,\lambda}\to\frac{1}{2}\sum^{\mathscr{R}}_{\lambda\in\mathbb{Z}_{\geq 1}}\varepsilon_{v,\lambda}=\frac{5}{24}-\frac{v}{4},$$

which is the Casimir energy of $\mathscr{B}(1,v)$. Thus, we have the regularized grand canonical partition function $\mathscr{Z}^{\mathscr{R}}_{\mathscr{B}(1,v)}(q,t)$ of $\mathscr{B}(1,v)$ such that

$$\mathscr{Z}^{\mathscr{R}}_{\mathscr{B}(1,v)}(q,t):=\mathscr{Z}_{B(1,v)}(q,t)q^{\frac{1}{2}\sum^{\mathscr{R}}_{\lambda\in\mathbb{Z}_{\geq 1}}\varepsilon_{v,\lambda}}=\mathscr{Z}_{\gamma^{\mathscr{R}}_{b,v},q}(t) \tag{19.2.4}$$

by equation (1.11.4). In particular, this $\mathscr{Z}^{\mathscr{R}}_{\mathscr{B}(1,v)}(q,t)$ is non-trivial.

19.2.2 Ideal fermion gases

Let us take the following with zero-point energies.

Definition 19.2. *For $v\in\mathbb{Q}$, let $\mathscr{H}_{f,v}$ denote the Hamiltonian operator such that*

$$\begin{aligned}\mathscr{H}_{f,v}&=\sum_{\lambda\in\mathbb{Z}_{\geq 1}}\frac{\varepsilon_{v,\lambda}}{2}[a^\dagger_{f,\lambda},a_{f,\lambda}]\\&=\sum_{\lambda\in\mathbb{Z}_{\geq 1}}\left(\varepsilon_{v,\lambda}a^\dagger_{f,\lambda}a_{f,\lambda}-\frac{\varepsilon_{v,\lambda}}{2}\right).\end{aligned} \tag{19.2.5}$$

If $u\in\mathbb{Z}_{\geq 1}$, then let $\mathscr{H}_{f,v,u}$ and $N_{f,u}$ be the Hamiltonian and number operators such that

$$\begin{aligned}\mathscr{H}_{f,v,u}&=\sum_{\lambda\in[\![u]\!]}\left(\varepsilon_{v,\lambda}a^\dagger_{f,\lambda}a_{f,\lambda}-\frac{\varepsilon_{v,\lambda}}{2}\right),\\N_{f,u}&=\sum_{\lambda\in[\![u]\!]}a^\dagger_{f,\lambda}a_{f,\lambda}.\end{aligned}$$

Also, let $\gamma^{\mathscr{R}}_{f,v}=\left(\left(\frac{1}{2},\frac{1}{2}-v,-\frac{5}{24}+\frac{v}{4}\right)\right)$.

Consider the fermion system $\mathscr{F}(1,v)$ of $\mathscr{H}_{f,v}$ and N_b, which extends $F(1,v)$ by the zero-point energies $-\frac{\varepsilon_{v,\lambda}}{2}$ in equation (19.2.5). Also, $\mathscr{F}(1,v)$ suggests the grand canonical partition function

$$\mathscr{Z}_{\mathscr{F}(1,v)}(q,t) = \mathrm{Tr}\left(e^{-\beta \mathscr{H}_{f,v}} \cdot e^{-\mu' N_f}\right).$$

However, this has $q^{-\frac{\sum_{\lambda\in\mathbb{Z}_{\geq 1}} \varepsilon_{v,\lambda}}{2}}$, which would make $\frac{1}{0}$. Hence, we consider the regularized grand canonical partition function $\mathscr{Z}^{\mathscr{R}}_{\mathscr{F}(1,v)}(q,t)$ of $\mathscr{F}(1,v)$ by the Ramanujan summation and a monomial convolution as follows.

For each $u \in \mathbb{Z}_{\geq 1}$, there is the fermion system $\mathscr{F}(1,v,u)$ of $\mathscr{H}_{f,v,u}$ and $N_{f,u}$. Also, $\lambda \in \mathbb{Z}_{\geq 1}$ and $n_\lambda \in \{0,1\}$ imply

$$\langle n_\lambda | e^{-\beta \varepsilon_{v,\lambda}\left(a^\dagger_\lambda \alpha_\lambda - \frac{1}{2}\right)} \cdot e^{-\mu' a^\dagger_\lambda \alpha_\lambda} | n_\lambda\rangle = e^{-\beta \varepsilon_{v,\lambda} n_\lambda} \cdot e^{-\mu' n_\lambda} \cdot e^{-\beta \cdot \frac{-\varepsilon_{v,\lambda}}{2}}.$$

This gives the grand canonical partition function of $\mathscr{F}(1,v,u)$:

$$\mathscr{Z}_{\mathscr{F}(1,v,u)}(q,t) = \mathrm{Tr}\left(e^{-\beta \mathscr{H}_{f,v,u}} \cdot e^{-\mu' N_{f,u}}\right) = (-q^{1-v}t;q)_u \cdot q^{\frac{-1}{2}\sum_{\lambda\in[\![u]\!]} \varepsilon_{v,\lambda}}.$$

Then, the Ramanujan summation (19.2.1) gives the regularization

$$\lim_{u\to\infty} -\frac{1}{2}\sum_{\lambda\in[\![u]\!]} \varepsilon_{v,\lambda} \to -\frac{1}{2}\sum^{\mathscr{R}}_{\lambda\in\mathbb{Z}_{\geq 1}} \varepsilon_{v,\lambda} = -\frac{5}{24} + \frac{v}{4},$$

which is the Casimir energy of $\mathscr{F}(1,v)$. Hence, we have the regularized grand canonical partition function $\mathscr{Z}^{\mathscr{R}}_{\mathscr{F}(1,v)}(q,t)$ of $\mathscr{F}(1,v)$ such that

$$\mathscr{Z}^{\mathscr{R}}_{\mathscr{F}(1,v)}(q,t) := \mathscr{Z}_{F(1,v)}(q,t) q^{-\frac{1}{2}\sum^{\mathscr{R}}_{\lambda\in\mathbb{Z}_{\geq 1}} \varepsilon_{v,\lambda}} = \mathscr{Z}_{\gamma^{\mathscr{R}}_{f,v},q}(t) \tag{19.2.6}$$

by equation (1.11.5). In particular, this $\mathscr{Z}^{\mathscr{R}}_{\mathscr{F}(1,v)}(q,t)$ is mathematically defined.

19.2.3 Ideal boson–fermion gases

Suppose integers $0 \leq d_1 \leq d_2$ such that $d_2 \in \mathbb{Z}_{\geq 1}$ and $\kappa \in \mathbb{Q}^{d_2}$. Consider the boson–fermion system $\mathscr{M}_{d_1,d_2}(\kappa)$ that has sub-systems $\mathscr{B}(1,\kappa_\lambda)$ for $\lambda \in [\![d_1]\!]$ and sub-systems $\mathscr{F}(1,\kappa_\lambda)$ for $\lambda \in [\![d_1+1,d_2]\!]$ with negligible interactions among sub-systems. Also, $\mathscr{M}_{d_1,d_2}(\kappa)$ proposes the grand canonical partition function:

$$\mathscr{Z}_{\mathscr{M}_{d_1,d_2}(\kappa)}(q,t) = \prod_{\lambda\in[\![d_1]\!]} \mathrm{Tr}\left(e^{-\beta \mathscr{H}_{b,\kappa_\lambda}} \cdot e^{-\mu' N_b}\right) \cdot \prod_{\lambda\in[\![d_1+1,d_2]\!]} \mathrm{Tr}\left(e^{-\beta \mathscr{H}_{f,\kappa_\lambda}} \cdot e^{-\mu' N_f}\right),$$

which have divergent zero-point energy sums. However, equations (19.2.4) and (19.2.6) give the regularized grand canonical partition function $\mathscr{Z}^{\mathscr{R}}_{\mathscr{M}_{d_1,d_2}(\kappa)}(q,t)$ of $\mathscr{M}_{d_1,d_2}(\kappa)$ such that

$$\mathscr{Z}^{\mathscr{R}}_{\mathscr{M}_{d_1,d_2}(\kappa)}(q,t) := \prod_{\lambda\in[\![d_1]\!]} \mathscr{Z}^{\mathscr{R}}_{\mathscr{B}(1,\kappa_\lambda)}(q,t) \cdot \prod_{\lambda\in[\![d_1+1,d_2]\!]} \mathscr{Z}^{\mathscr{R}}_{\mathscr{F}(1,\kappa_\lambda)}(q,t),$$

which is mathematically defined and non-trivial.

Furthermore, $\alpha = \iota^{d_2}(1)$ yields the multimonomial index $\left(d_2, w, \alpha, \gamma^{\mathscr{R}}_{\kappa}\right)$ such that

$$\gamma^{\mathscr{R}}_{\kappa,\lambda} = \begin{cases} \gamma^{\mathscr{R}}_{b,\kappa_\lambda} \text{ if } \lambda \in [\![d_1]\!], \\ \gamma^{\mathscr{R}}_{f,\kappa_\lambda} \text{ if } \lambda \in [\![d_1+1, d_2]\!]. \end{cases}$$

By equations (19.2.4) and (19.2.6), we obtain the monomial convolution

$$\mathscr{Z}^{\mathscr{R}}_{\mathscr{M}_{d_1,d_2}(\kappa)}(q,t) = \mathscr{M}(d_2, w, \alpha, \gamma^{\mathscr{R}}_{\kappa}, q, t). \tag{19.2.7}$$

Also, $\mathscr{M}_{d_1,d_2}(\kappa)$ has the Casimir energy

$$\sum_{\lambda \in [\![d_1]\!]} \left(-\frac{5}{24} + \frac{\kappa_\lambda}{4}\right) + \sum_{\lambda \in [\![d_1+1,d_2]\!]} \left(\frac{5}{24} - \frac{\kappa_\lambda}{4}\right) = -\frac{5(2d_1 - d_2)}{24} + \frac{\sum \kappa[1:d_1] - \sum \kappa[d_1+1:d_2]}{4}.$$

Thus, we obtain

$$\mathscr{Z}^{\mathscr{R}}_{\mathscr{M}_{d_1,d_2}(\kappa)}(q,t) = \mathscr{Z}_{M_{d_1,d_2}(\kappa)}(q,t) \cdot q^{-\frac{5(2d_1-d_2)}{24} + \frac{\sum \kappa[1:d_1] - \sum \kappa[d_1+1:d_2]}{4}}. \tag{19.2.8}$$

Now, equations (19.2.7) and (19.2.8) give the explicit description of $\mathscr{Z}^{\mathscr{R}}_{\mathscr{M}_{d_1,d_2}(\kappa)}(q,t)$ by q-multinomial coefficients in Claim 1 of Proposition 18.13 (see Proposition 18.7 for eta products when $\kappa_\lambda = 1$ for each $\lambda \in [\![d_2]\!]$).

19.3 Phase transitions of ideal boson–fermion gases

As in Section 1.11, the t-power series of $\mathscr{Z}_{M_{d_1,d_2}(\kappa)}(q,t)$ realize the Helmholtz free energies of the ideal boson–fermion systems $M_{d_1,d_2}(\kappa)$ without the Casimir energies. Hence, the systems $M_{d_1,d_2}(\kappa)$ obtain statistical-mechanical phase transitions by the merged-log-concavity in Proposition 17.20. The same holds for $\mathscr{M}_{d_1,d_2}(\kappa)$ with the Casimir energies in equation (19.2.8).

In particular, inequalities (17.3.1) and (17.3.2) that give phase transitions in Proposition 17.20 imply $1 - v > 0$ and $\frac{1}{2} - v > -\frac{1}{2}$ in equations (1.11.4) and (1.11.5). Therefore, we obtain the energy positivity

$$\varepsilon_{v,\lambda} = \lambda - v > 0$$

for $\lambda \in \mathbb{Z}_{\geq 1}$ in the Hamiltonians $H_{b,v}$ and $H_{f,v}$ by the phase transitions.

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